THE UNIVERSITY OF IOWA
Iowa City, Iowa

NOV 18 1969

PROJECT THEMIS

Vibration and Stability of Military Vehicles

OPTIMUM DESIGN OF SPATIAL STRUCTURES

D. L. Bartel
Department of Mechanics and Hydraulics
University of Iowa

This report is the Ph.D. Dissertation of the author,
done under the supervision of Professor K. Rim.

Project Themis - University of Iowa
"Vibration and Stability of Military and
Other Complex Vehicular Systems"
Contract No. DAAF03-69-C-0014

August 1969

ABSTRACT

This research presents a systematic approach to the optimal design of spatial structures for minimum weight subject to constraints on stress and geometry. The optimization procedures discussed are general and may be applied to structures which can be analyzed by matrix displacement or finite element methods.

Two methods of mathematical programming are applied to obtain a minimum weight design. The first is the sequential unconstrained minimization technique (SUMT), and the second is the method of constrained steepest descent with state equations (CSDS). Both of these techniques require derivatives of the objective and constraint functions to improve estimates of the optimum design. In many structural problems, it is very difficult or impossible to compute these derivatives exactly; existing structural analysis algorithms are generally not equipped to compute these derivatives. In order to take full advantage of existing analysis capability, the programming techniques in this research have been developed assuming that such derivatives are not available.

Optimal structural design problems are characterized by an objective function (the weight), state variables (the stresses and deflections), design variables, state equations (the structural analysis), and constraints which may be functions of the design and state variables. When the state equations are used to write all of the

constraints as functions of the design variables, a nonlinear programming problem results. The sequential unconstrained minimization technique reduces the constrained nonlinear programming problem to a sequence of unconstrained problems which can be solved using existing unconstrained minimization techniques. A SUMT program was written for this research using Powell's method of unconstrained minimization without derivatives. The required minimization of a function along a line uses a combination of a Fibonacci search (to bracket the minimum) and a quadratic approximation of the minimum.

The method of constrained steepest descent differs from the usual nonlinear programming problem in that the state equations and the state variable constraints appear explicitly in the formulation. This provides a natural matching of the essential features of the design problem and the method used to obtain its solution. The design problem is linearized about a candidate design and the desired improvement in the design variables, δx, is required to be small by demanding that $\delta x^T w^{-1} \delta x = \xi^2$, where ξ is a small number and w is a positive definite weighting matrix. The Kuhn-Tucker necessary conditions are then applied to the resulting nonlinear problem. As a direct consequence, δx is specified in terms of two components; δx_1 which reduces the objective function consistent with the constraints, and δx_2 which directs the search for a minimum back to the feasible region if constraints have been violated. The method was applied using both exact and approximate derivatives, so that its effectiveness when derivatives are not available could be assessed.

A spatial structure which occurs frequently in practice is the plane frame with out-of-plane loads. Although such structures are generally made up of relatively few members, they may have many design variables since several design parameters must be specified for each member. The programming methods were applied to a number of two and three member frames of this type. From the results, it appears that CSDS has significant advantages over SUMT both in terms of computational time and the number of times that candidate designs must be analyzed. The results also show that CSDS performs as well when derivatives are approximated as it does when they can be computed exactly. The effectiveness of SUMT is reduced significantly if the derivatives are unavailable.

Abstract approved: _Kevan Rian_, dissertation supervisor

Professor
Mechanics & Hyde, title and department

July 16, 1969 date

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

LIST OF SYMBOLS

t	wall thickness of a hollow member
b	width of a member
h	height of a member
q	generalized coordinates in structural analysis
$\tilde{V}(q)$	total potential energy
$U(q)$	strain energy
$\Omega(q)$	potential energy of external loads
A	matrix used in structural analysis, unless otherwise specified
Q	vector of joint displacements
P	vector of generalized forces unless otherwise specified
M	vector of member end reactions
F	vector of "fixed end" moments and loads
C	matrix used to compute stresses unless otherwise specified
$\tilde{P}$	vector involving external loads
Z	vector of state variables
$\tilde{A}$	coefficient matrix for state equations
$\hat{P}$	right-hand side of state equations
x	vector of design variables
$V(x)$	volume of the frame
σ	normal stress

τ	shear stress
σ_f	failure stress
σ_{max}	maximum allowable failure stress
$\psi(z)$	vector-valued constraint function of the state variables
x_{min}	vector of minimum allowable design variables
x_{max}	vector of maximum allowable design variables
$\phi(x)$	vector-valued constraint function of the design variables
n	number of design variables
ℓ	number of ϕ constraints
m	number of ψ constraints, unless otherwise specified
k	number of state variables, unless otherwise specified
$h(z,x)$	vector-valued function of state and design variables
R	feasible region in design space
$g(x)$	vector-valued constraint function in the nonlinear programming problem
r_k	single scalar variable used in SUMT
$U(x,r_k)$	unconstrained function to be minimized in SUMT
$S(r_k)$	scalar-valued function used in SUMT
$I(x)$	scalar-valued function used in SUMT
ξ_i	direction vector in n space

p_i	estimate of the minimum in Powell's method
ε_{sx}	convergence criterion in SUMT
ε_p	convergence criterion in Powell's method
ε_L	convergence criterion for minimization along a line
$H, \hat{H}$	vector-valued function in Kuhn-Tucker necessary conditions
$\lambda, \mu, \nu, \zeta,$ $\tilde{\mu}, \tilde{\nu}, \bar{\lambda}, \bar{\nu}$	generalized Lagrange multipliers
$\lambda^\phi, \lambda^\psi, \Lambda^\phi, \Lambda^\psi$	matrices used to define the linearized problem in CSDS
λ^J, Λ^J	vectors used to define the linearized problem in CSDS
$\tilde{\phi}$	vector of ϕ constraints which are violated or identically zero
$\tilde{\psi}$	vector of ψ constraints which are violated or identically zero
$\Delta\tilde{\phi}$	vector of changes demanded in $\tilde{\phi}$
$\Delta\tilde{\psi}$	vector of changes demanded in $\tilde{\psi}$
δx	small change in the design variables
δz	small change in the state variables
$\delta x_1, \delta x_2$	component vectors of δx
η	constant determining magnitude and direction of δx
η_{max}	maximum allowable value of η
ε_x	convergence criterion for CSDS

CHAPTER I

INTRODUCTION

1.1 The Class of Problems to be Solved

The subject of optimum structural design has received much attention in recent years. The optimal design of simple structural elements and structures consisting of these elements has been studied extensively. A number of investigators have considered plane frames and trusses, but spatial frames have received little attention. It is to this topic that this research is directed. The problem to be solved is the minimum weight design of a spatial frame subject to constraints on stress and geometry.

One type of spatial structure that occurs frequently in practice is the plane frame with out-of-plane loads. Similar frames are often found in automotive, construction vehicle, and agricultural equipment applications. Real design problems of this type initiated this research and are used as example problems for the solution techniques investigated. Although such structures are generally made up of relatively few members, they may have many design variables since several design parameters must be specified for each member. In addition these frames are often required to support or transmit loads at many points in the structure. When such frames are mass produced the design which requires the least material has a significant economic advantage.

The design must also satisfy constraints on stresses and geometry. In most cases the structure will fail by fatigue; therefore, the maximum stresses in the frame must be restricted to values well below the elastic limit of the material. Limits on the design variables of the frame may be dictated by current manufacturing capabilities or other related limitations. It is apparent from this discussion that structural design problems, like almost all design problems, have a merit or objective function, a set of design variables, a set of state variables, such as stresses or deflections which describe the behavior of the frame, a set of state equations which determine the state at a given design point, constraints on the design variables, and constraints on the state variables.

Two methods of mathematical programming are applied to obtain a minimum weight design. The first is the sequential unconstrained minimization technique, and the second is a constrained steepest descent method which uses the state equations directly in the optimization process. Both of these techniques require derivatives of objective and constraint functions to predict better approximations to the optimum design. These derivatives may be cumbersome or impossible to compute exactly. In addition it is desirable in structural optimization to take advantage of existing analysis algorithms which in general yield only function values, not derivatives. In order to take full advantage of existing analysis capability, the programming techniques should be effective when only function values are available. The optimization procedures discussed in this research are general

and may be applied to spatial frames analyzed by matrix displacement or finite element methods.

In Chapter II the method of structural analysis is discussed and the formulation of the optimization problem as a mathematical programming problem is presented. The sequential unconstrained minimization technique is outlined in Chapter III and the constrained steepest descent method with state equations is developed in Chapter IV. In Chapter V the application of the programming methods to some example problems is discussed and the results for these examples are presented. Conclusions concerning the results and the relative merit of the design methods are discussed in Chapter VI.

1.2 Literature Survey

The field of optimum structural design has been exhaustively surveyed from Galileo to the present in review papers by Wasiutynski and Brandt [24] and Sheu and Prager [22]. In addition a review by Gerard [9], which is particularly applicable to aerospace vehicles, contains numerous structural references. For extensive bibliographies covering all aspects of optimal design and evaluations of the current state of the art, the reader is referred to these articles.

In recent years, a number of investigators have applied the methods of nonlinear programming to optimum structural design. Schmit, Kicher, and Morrow [20] solved the problem of integrally stiffened waffle plates using a method of alternate steps. In this method steepest descent moves are made until a constraint is violated. The

step is then adjusted until the design is on or near a constraint.
This point is considered to be bound and an alternate step to a free
(unconstrained) point is sought. The technique uses a random number
generator to create proposed alternate step designs which have the
same weight as the current bound design and which do not violate addi-
tional constraints. Gellatly and Gallagher [7] presented the theo-
retical basis for a design procedure which includes steepest descent
and alternate step moves similar to those of Schmit, et. al. [20].
They apply this procedure to the design of plane trusses and stiffened
panels [8]. The objective function is linear in both cases, since
only one design variable per structural element is considered.

Others have also shown that optimum structural design can be
formulated as a nonlinear programming problem. Pope [15] solved plane
trusses by reducing the resulting nonlinear programming problem to a
sequence of linear programming problems, and Best [2] suggested the
use of a gradient projection technique. Recently Brown and Ang [3]
applied the gradient projection method to the elastic design of WF
steel frames. A problem involving four design variables and eleven
constraints was solved and forward finite difference approximations
were used to evaluate the derivatives required.

A somewhat different approach was suggested by Schmit and Fox
[21]. Instead of developing designs, analyzing them, and using infor-
mation from the objective function and the constraints to predict
improved designs, a special function, ψ, is constructed and minimized.
The ψ function contains among other terms the weight (or other

quantity to be minimized) and an estimate of the minimum weight. The function ψ is so structured that when $\psi = 0$ all constraints and the equations of equilibrium and compatibility are satisfied. In addition the weight is decreased to the estimated minimum. The estimate of the minimum weight is then reduced and the process is repeated until the weight cannot be further decreased. At each step the solution $\psi = 0$ is found by methods for unconstrained minimization.

A method of nonlinear programming known as the sequential unconstrained minimization technique (SUMT), has also received some attention. Nicholls [14] used the method to solve some plane truss problems, but had limited success. The lack of success was attributed to the use of a first order gradient technique for the unconstrained minimizations. Brown and Ang [4] have used SUMT to obtain starting values for the gradient projection method and have noted the possibility of solving structural design problems using SUMT instead of other programming techniques. A recent book by Bracken and McCormick [1] discusses examples of nonlinear programming problems to which SUMT has been successfully applied. Among these is the design of a vertically corrugated transverse bulkhead for an oil tanker. The design is specified by six design variables and must satisfy sixteen constraints. The objective function is nonlinear and there are both linear and nonlinear constraints. Derivatives of the constraint functions were available without approximation.

The programming problems associated with optimum design are often nonconvex. In general, therefore, a local minimum is obtained.

Usually widely separated starting values are chosen and if the method converges to the same point each time, this point is taken as the global minimum. Toakley [23] discusses the problem of the global optimum for statically determinate plane trusses. A change of variable is introduced to obtain a convex nonlinear programming problem which is then solved by reducing it to a sequence of linear programming problems.

The literature concerning mathematical programming has also been voluminous in the past several years. Recent books by Wilde and Beightler [25], and Saaty and Bram [19] discuss a wide variety of optimization techniques. The former treats all types of problems while the latter has an extensive section on the methods of nonlinear programming. A book by Fiacco and McCormick [5] presents the theoretical basis of the sequential unconstrained minimization technique. In addition, these books serve as a review and bibliography of all of the major contributions to the field.

Fletcher's review [6] of unconstrained minimization techniques, which do not require derivatives, is particularly pertinent to this research. It suggests that, of the methods available, the one due to Powell [16] based on conjugate directions is the most effective. Wortman [25] has written a program which combines SUMT and Powell's method. He reported extreme sensitivity to the starting values and proposed that the technique could be used to determine binding constraints. These constraints could then be used to decrease the

dimension of the optimization problem and SUMT could then be reapplied. No results of this procedure were included in the report.

The growing literature available on SUMT [5 and 1] indicates that the method is most successful when derivatives of both the objective and constraint functions can be computed exactly. One of the objects of this dissertation is to consider the applicability of SUMT to structural design problems when derivatives are not available.

As stated previously, almost all design problems have an objective function, design variables, state variables, a set of state equations, state constraints, and design constraints. The nonlinear programming approaches cited above (except for Schmit and Fox [21]), use the state equations indirectly to express all of the constraints as functions of the optimization variables. The method of constrained steepest descent with state equations, to be developed in Chapter IV, differs from these methods in that the state equations and state variable constraints appear explicitly in the mathematical programming problem. This provides a very natural matching of the essential features of the design problem and the method used to obtain its solution.

CHAPTER II

FORMULATION OF THE PROBLEM

2.1 The Problem

This chapter presents a general formulation for the optimal
design of minimum weight spatial structures, subjected to geometrical
as well as stress constraints. The geometrical constraints are
restrictions placed on the dimensions of the structure, whereas the
stress constraints correspond to a failure criterion established by
one of the failure theories. The elements of such a structure may
undergo extension, bending, twisting, and shear deformation. The
formulation of the design problem in this chapter is general, but for
clarity in presentation, some examples which are characteristic of
the general class of problems will be used in the development.

Consider a structure which consists of rectangular, hollow beams.
These beams are mutually perpendicular to each other and are joined
at their ends by rigid joints so that forces, bending moments, and
twisting moments are transmitted from member to member. The structure
is loaded by concentrated forces located arbitrarily along each of
the members. The frame shown in Figure 2.1 is an example of this
type of structure; Figure 2.2 shows a typical member cross-section.
Bending and twisting effects in each member will be considered but
shear due to transverse loads and axial deformation of the members
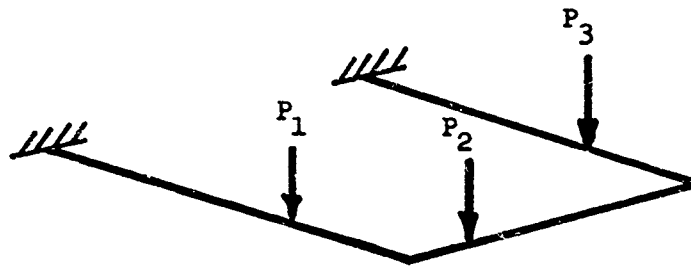
FIGURE 2.1   Three Member Frame with Out-of-Plane Loads
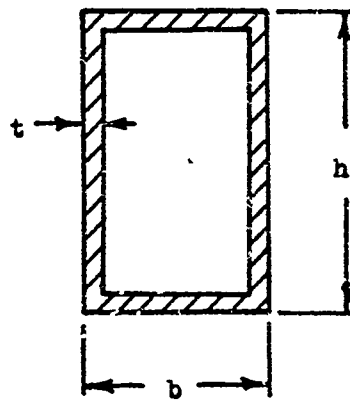
FIGURE 2.2   Typical Member Cross-Section

will be neglected. The thickness, t, of the hollow sections is constant and all members are assumed to be made of the same linearly elastic material.

The design variables in this problem are the thickness, t, width, b, and height, h, of the individual members (see Figure 2.2). These are to be chosen so that the objective function (the weight of the structure) is a minimum and the constraints on stress and geometry are satisfied. The state variables, describing the behavior of the structure for a given design, are the stresses at critical points in the frame and the deflections at the joints. The state equations which determine the state variables are the equations of structural analysis which are discussed in the next section.

2.2 The State Equations

The state variables are determined from a set of matrix equations which are derived from energy principles. Matrix methods for the analysis of structures have received much attention in recent literature. Books devoted to this subject include those by Rubenstein [18], Przemieniecki [17], and Zienkiewicz [27]. The analysis of the structures investigated in this research follows a method outlined by Langhaar [11]. When this method is written in matrix notation, the resulting equations are of the same form as those used in the finite element and matrix displacement methods of structural analysis. Details of the analysis are given in the appendix; the principal features of the method are as follows.

The total potential energy of a structure and the applied loads may be written as

$$\tilde{V}(q) = U(q) + \Omega(q) \qquad (2\text{-}1)$$

where $U(q)$ is the strain energy of deformation, $\Omega(q)$ is the potential energy of the external loads, and q is a vector whose elements are the cartesian components of the independent kinematically admissible angular and linear joint displacements. For sufficiently small displacements q_i, the internal strain energy $U(q)$ is a positive definite quadratic form in the generalized coordinates q_i,

$$U = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} q_i q_j. \qquad (2\text{-}2)$$

It is shown in Langhaar's book [11] that the required conditions for equilibrium are

$$\frac{\partial \tilde{V}}{\partial q_i} = 0 \qquad i = 1, 2, \ldots, n, \qquad (2\text{-}3)$$

where n is the number of generalized coordinates. This principle of stationary potential energy applied to equation (2-1) yields

$$\frac{\partial U}{\partial q_i} + \frac{\partial \Omega}{\partial q_i} = 0 \qquad i = 1, 2, \ldots, n. \qquad (2\text{-}4)$$

Using the definition of the components of generalized force given by

$$P_j = -\frac{\partial \Omega}{\partial q_i}. \qquad (2\text{-}5)$$

the following set of equations is obtained

$$\sum_{j=1}^{n} a_{ij} q_j = P_i, \qquad i = 1, 2, \ldots, n. \qquad (2\text{-}6)$$

In matrix notation these equations may be expressed as

$$AQ = P \qquad (2\text{-}7)$$

which corresponds directly to the basic equation of the matrix displacement method of structural analysis [17].

The elements of matrix A are known functions of the design variables and the elements of vector P are functions of the external loads. Equation (2-7) may be solved for the joint displacements Q. In the appendix it is shown that the end reactions and the member stresses may also be written as matrix equations. For end reactions the following equation is obtained

$$M = BQ + F, \qquad (2-8)$$

in which M is the vector of end reactions, B is a matrix whose elements are functions of the design variables, Q is the vector of joint displacements, and F is a vector of the "fixed-end" moments and loads.

The stresses may be computed from the following equation

$$S = CM + \tilde{P} \qquad (2-9)$$

where S is the vector of stresses calculated at critical points in the structure (see Appendix), C is a matrix whose elements are functions of the design variables, M is the vector of end reactions, and P is a vector whose elements are functions both of the applied loads and the design variables. By combining equations (2-8) and (2-9), the following equation is obtained

$$S = CBQ + CF + \tilde{P}. \qquad (2-10)$$

The analysis of the structure may then be summarized by the following equations

$$AQ = P$$

$$M = BQ + F$$

and $$S = CBQ + CF + \tilde{P}. \qquad (2-11)$$

The fundamental equations of the finite element technique are given by Zienkiewicz [27] and are of the same form as equations (2-11). Conceptually the optimization procedures investigated in this research can then be applied to the broad classes of problems which can be analyzed by matrix displacement and finite element techniques. Equations (2-11) apply to linearly elastic structures under the action of conservative external loads.

Physically the behavior of the frame is most often described by the joint deflections and stresses. Therefore it is natural to define these variables as the state variables, Z,

$$Z = \begin{bmatrix} Q \\ S \end{bmatrix}. \tag{2-12}$$

Given the external loads and a set of design variables, the first and third equations of set (2-11) can then be combined to yield one matrix equation which determines Z. This equation is given by

$$\tilde{A}Z = \hat{P}, \tag{2-13}$$

where

$$\tilde{A} = \begin{bmatrix} A & 0 \\ -CB & I \end{bmatrix}$$

and

$$\hat{P} = \begin{bmatrix} P \\ CF+\tilde{P} \end{bmatrix}.$$

The analysis of the structure (determination of the state variables) is therefore reduced to the solution of a single set of simultaneous equations.

2.3 The Objective Function

Since all members are assumed to be made of the same material, the weight of the frame is the product of a single weight density, and the volume of the frame. Therefore the problem of determining the minimum weight is equivalent to that of determining the minimum volume. The volume of the ith member of the frame is given by

$$V_i = [b_i h_i - (b_i - 2t_i)(h_i - 2t_i)]C_i$$

or

$$V_i = [2t_i b_i + 2t_i h_i - 4t_i^2]C_i \qquad (2\text{-}14)$$

where t_i, b_i, and h_i are defined in Figure 2.2, and C_i is the length of the member. The design variables are systematically assigned as follows, m being in this case the number of frame members,

$$x_1 = t_1$$

$$x_2 = b_1$$

$$x_3 = h_1$$

$$\vdots$$

$$x_{3i-2} = t_i$$

$$x_{3i-1} = b_i$$

$$x_{3i} = h_i \qquad i = 1, \ldots, m.$$

Using this notation, equation (2-14) can be written in terms of the design variables:

$$V_i(x) = [2x_{3i-2}x_{3i-1} + 2x_{3i-2}x_{3i} - 4x_{3i-2}^2]C_i. \quad (2\text{-}15)$$

The volume of the frame is obtained by summing the volumes of the individual members and is a quadratic in the design variables.

$$V(x) = \sum_{i=1}^{m} V_i(x) \tag{2-16}$$

2.4 The Constraints

Structural failure is said to occur when the state variables or functions of the state variables exceed certain preassigned limits. In space frames failure may be due to either excessive stresses or deflections. In the typical structures being considered, failure is assumed to occur when the combination of normal stress due to bending and shear stress due to torsion exceeds a maximum limit. The stresses were combined according to the distortion energy theory of failure. The failure stress σ_f is computed as follows:

$$\sigma_f = \sqrt{\sigma^2 + 3\tau^2} \tag{2-17}$$

where σ is the normal stress and τ is the shearing stress at a critical point of the structure. The statement that the failure stress must not exceed a specified maximum is written

$$\sqrt{\sigma^2 - 3\tau^2} - \sigma_{max} \leq 0 \tag{2-18}$$

and the general form of a state variable constraint is

$$\psi(z) \leq 0. \tag{2-19}$$

The choice of design variables may also be limited. These limits may occur because of space restrictions or result from limits imposed

by current manufacturing facilities. Constraints of this type are introduced into the structural design problem of interest by requiring that the design variables satisfy the following inequality:

$$x_{min} \leq x \leq x_{max}. \tag{2-20}$$

This inequality is equivalent to two constraint relationships, namely,

$$x_{min} - x \leq 0 \tag{2-21}$$

and

$$x - x_{max} \leq 0 \tag{2-22}$$

The general form of these constraints is

$$\phi(x) \leq 0. \tag{2-23}$$

2.5 The Mathematical Programming Problem

The state equations, the objective function, and the constraints associated with the optimal structural design problem have been defined in the previous sections of this chapter. These functions may be used to define the following mathematical programming problem:

$$\text{minimize } V(x) \tag{2-24}$$

subject to

$$\phi_i(x) \leq 0, \qquad i = 1, \ldots, \ell$$

$$\Psi_i(z) \leq 0, \qquad i = 1, \ldots, m$$

and $\qquad h_i(z,x) = 0, \qquad i = 1, \ldots, k.$

The equality constraints in problem (2-24) are the state equations (2-13). The most common approach to this problem has been to use the state equations to write the state constraints as functions of the design variables. This results in the following nonlinear

programming problem, which may be attacked by several available tech-
niques (see Section 1.2),

$$\text{minimize } V(x) \qquad\qquad (2\text{-}25)$$

subject to

$$g_j(x) \leq 0 \qquad\qquad j = 1, \ldots, m+\ell$$

where

$$g(x) = \begin{bmatrix} \phi(x) \\ \psi(x) \end{bmatrix}.$$

A design which satisfies all of the impose constraint conditions is
called a feasible design. The constraints define a feasible region,
R, in design space,

$$R = \{x: \quad g(x) \leq 0\}. \qquad\qquad (2\text{-}26)$$

If the nonlinear programming problem (2-25) is convex, then it can be
shown that the solution attained is a global optimum. It will now be
shown that the structural optimization problem formulated in this
chapter is nonconvex.

The convexity of sets and functions may be investigated by using
three theorems which are proven in [19].

Theorem 2.1 The set of points R which satisfy a
constraint $g(x) \leq 0$, where $g(x)$ is a convex func-
tion is a convex set.

Theorem 2.2 The intersection R of a family F of
convex sets is a convex set.

Theorem 2.3 If $V(x)$ is a twice differentiable

function in an open convex set R, it is convex

in R if and only if the quadratic form

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 V}{\partial x_i \partial x_j} \lambda_i \lambda_j$$

is positive-semidefinite for every point x in R.

Because the state variable constraints are complicated functions

of the design variables, it is difficult to apply Theorem 2.3 to

these functions. Therefore one cannot immediately determine whether

or not the feasible region, R, is a convex set. Consequently, the

convexity of the objective function will be investigated by applying

the criterion of Theorem 2.3 assuming that the region R satisfies the

conditions of the theorem.

Consider the volume of the three member frame shown in Figure 2.1

$$V(x) = (2x_1 x_2 + 2x_1 x_3 - 4x_1^2)C_1$$

$$+ (2x_4 x_5 + 2x_4 x_6 - 4x_4^2)C_2$$

$$+ (2x_7 x_8 + 2x_7 x_9 - 4x_7^2)C_3. \tag{2-27}$$

The matrix of second derivatives required in Theorem 2.3 is

$$
\begin{bmatrix}
-8c_1 & 2c_1 & 2c_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
2c_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2c_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -8c_2 & 2c_2 & 2c_2 & 0 & 0 & 0 \\
0 & 0 & 0 & 2c_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2c_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -8c_3 & 2c_3 & 2c_3 \\
0 & 0 & 0 & 0 & 0 & 0 & 2c_3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2c_3 & 0 & 0
\end{bmatrix} .
$$

The analysis of quadratic forms, outlined in Langhaar's book [11], may be applied to show that the quadratic form based on this matrix is indefinite. Consequently, by Theorem 2.3, $V(x)$ is a nonconvex function when the region R is convex. Therefore problem (2-25) is a nonconvex programming problem.

The most that can be guaranteed for a nonconvex problem is that its solution is a local minimum. The most often used technique for seeking the global optimum in this situation is to attempt to determine all of the local minima by starting from many widely separated initial points.

In summary the optimal structural design problem has been formulated as a mathematical programming problem in two different ways. In one case the constraints are considered to be functions of the design variables alone. The sequential unconstrained minimization technique will be applied to this problem in Chapter III. In Chapter IV the constrained steepest descent method with state equations, will

be applied to problem (2-24). This formulation considers constraints which are functions of both the state variables and the design variables, and uses the state equations directly in the solution.

CHAPTER III

THE SEQUENTIAL UNCONSTRAINED MINIMIZATION TECHNIQUE

3.1 General Discussion

In Chapter II it was shown that the optimum design of a spatial frame could be formulated in the following way.

$$\text{minimize} \quad v(x)$$

subject to

$$g_i(x) \leq 0 \qquad i = 1, 2, \ldots, m+\ell$$

(3-1)

The constraints are complicated nonlinear functions of the design variables. Because of this, the direct handling of these constraints in the nonlinear programming problem can be difficult. The sequential unconstrained minimization technique (SUMT) handles these constraints indirectly and has been used successfully on problems of this type. The theoretical basis of this technique, as well as helpful suggestions for computation, may be found in the recent book by Fiacco and McCormick [5]. There are several versions of SUMT which may be applied to given problems depending upon the nature of the objective function and the constraints. For problem (3-1) the interior method should be used. In this method the quest for a minimum is always carried out within the feasible region and strict equality constraints are not allowed.

The method is applied by augmenting the objective function to define a new function $U(x, r_k)$,

$$U(x, r_k) = V(x) + S(r_k)I(x). \qquad (3\text{-}2)$$

The number r_k is always positive. $I(x)$ is a scalar function of the design variables and is continuous in the feasible region R, where R is defined by

$$R = \{x: \ g_i(x) < 0, \quad i = 1, 2, \ldots, m+\ell\}.$$

$I(x)$ also has the property that if $\{x_k\}$ is any infinite sequence of points in R converging to x_B such that $g_i(x_B) = 0$ for at least one i, then

$$\lim_{k \to \infty} I(x_k) = +\infty.$$

$S(r_k)$ is a scalar-valued function of the single variable r_k, and has the following properties. If $r_1 > r_2 > 0$, then $S(r_1) > S(r_2) > 0$, and if $\{r_k\}$ is an infinite sequence of points such that

$$\lim_{k \to \infty} r_k = 0$$

then

$$\lim_{k \to \infty} S(r_k) = 0.$$

These properties are basic to the convergence proofs which may be found in Fiacco and McCormick [5].

The most common forms for $S(r_k)$ and $I(x)$ are the following:

$$S(r_k) = r \qquad (3\text{-}3)$$

$$I(x) = -\sum_{i=1}^{m+\ell} \frac{1}{g_i(x)}. \qquad (3\text{-}4)$$

Using these, the function $U(x, r_k)$ becomes

$$U(x,r_k) = V(x) - r \sum_{i=1}^{m+\ell} \frac{1}{g_i(x)}, \qquad g_i(x) < 0. \qquad (3\text{-}5)$$

The penalty term provides a steep gradient at the boundary of the feasible region so that when the function $U(x,r_k)$ is minimized the search is kept in the feasible region R. When $g_i(x) \geq 0$, $U(x,r_k) = K$, where K is a large positive constant. This avoids difficulties which may arise in the unconstrained minimization if a constraint is inadvertently violated.

The algorithm for finding the constrained minimum of a function using interior SUMT is as follows.

(1) Choose r_1 and an estimate of the minimum $x^{(0)}$ interior to R.

(2) Determine the unconstrained minimum $x(r_1)$ of $U(x,r_1)$ in the feasible region R.

(3) Use $x(r_1)$ as a new starting point to determine the minimum $x(r_2)$ of $U(x,r_2)$ where $r_1 > r_2 > 0$.

(4) Continue the process, finding the local minimum of $U(x,r_k)$ starting from $x(r_{k-1})$ for a strictly monotonically decreasing sequence $\{r_k\}$.

Fiacco and McCormick show that if $V(x)$ and $g_i(x)$, $i = 1, 2, \ldots, m+\ell$ are continuous, if $I(x)$ and $S(r_k)$ satisfy the conditions stated previously, if there exists a relative minimum $\bar{x}$ in R such that $f(\bar{x}) > f(x)$ for all $x \neq \bar{x}$ in some neighborhood of $\bar{x}$, and $\bar{x}$ is not an isolated point in R, and if $\{r_k\}$ is a monotone decreasing sequence which converges to zero, then the sequence of local unconstrained minima

obtained in the above procedure exists and converges to a local minimum of problem (3-1).

The use of SUMT is appealing in that existing unconstrained minimization techniques can be used to obtain the minimum for each $U(x, r_k)$. The method, therefore, provides a means of reducing the value of the objective function without violating the constraints. Furthermore it is not necessary to involve the constraints directly or to move along them toward the constrained minimum of the objective function. This is a distinct advantage when the constraints are complicated nonlinear functions of the design variables as they are in the class of problems under consideration.

In the next section an unconstrained minimization technique which does not require derivatives will be discussed. Acceleration of convergence by extrapolation will be discussed in Section 3.3, and other computational considerations will be presented in Section 3.4. An evaluation of the method as applied to the optimum structural design problem may be found in Chapter V.

3.2 Finding an Unconstrained Minimum

As stated in Chapter I, one of the goals of this research is to investigate techniques that will be effective when derivatives of the objective and constraint functions are cumbersome or impossible to calculate. The method used to determine the unconstrained minimum of $U(x, r_k)$ in SUMT must therefore be able to find a minimum using only function values.

The earlier techniques such as tabulation, random search, or improving one variable at a time are basically inefficient and unreliable. The first improvements of these methods were based on ad hoc procedures. Only recently efficient techniques have been introduced which are based on successive minimizations along conjugate directions.

In his review of minimization techniques that do not require the calculation of derivatives, Fletcher [6] suggests that of the methods available, Powell's [16] is the most satisfactory. Powell's method requires fewer function evaluations than other techniques and has the advantage of quadratic convergence near the minimum. The method will find the minimum of a quadratic in a finite number of steps and converges to the minimum from an unfavorable starting point more efficiently than other available algorithms.

Powell's method is based on the minimization of a quadratic,

$$f(x) = x^T[A]x + bx + c. \tag{3-6}$$

Directions ξ_1 and ξ_2 are said to be conjugate with respect to A if

$$\xi_1^T[A]\xi_2 = 0. \tag{3-7}$$

Each iteration starts from the best previous estimate to the minimum p_0. Successive searches for a minimum are made along a set of linearly independent directions, ξ_1, ξ_2, ..., ξ_n. These directions are initially chosen as the coordinate directions so that the first iteration is identical to that of changing one parameter at a time. Each iteration generates a new direction ξ, and for the second iteration the set of linearly independent directions is chosen to be

ξ_2, ξ_3, ..., ξ_n, ξ. The new direction ξ is chosen so that if a quadratic is being minimized all the directions will be conjugate after n iterations. Powell proves that, as a direct consequence, the exact minimum of the quadratic is found. One iteration of the basic method is as follows.

(i) For $r = 1,2,...,n$ calculate λ_r so that $f(p_{r-1} + \lambda_r \xi_r)$ is a minimum and define $p_r = p_{r-1} + \lambda_r \xi_r$.

(ii) For $r = 1,2,...,n-1$ replace ξ_r by ξ_{r+1}.

(iii) Replace ξ_n by $(p_n - p_0)$.

(iv) Choose λ so that $f(p_n + \lambda\{p_n - p_0\})$ is a minimum and replace p_0 by $p_n + \lambda\{p_n - p_0\}$.

The basic procedure may be unstable for non-quadratic functions because it tends to choose nearly dependent directions. Powell shows that this difficulty can be eliminated by using the following procedure which allows a direction other than ξ_1 to be discarded and under some conditions uses the old set of linearly independent directions again.

(i) For $r = 1,2,...,n$ calculate λ_r so that $f(p_{r-1} + \lambda_r \xi_r)$ is a minimum and define $p_r = p_{r-1} + \lambda_r \xi_r$.

(ii) Find the integer m, $1 \leq m \leq n$, so that $\{f(p_{m-1}) - f(p_m)\}$ is a maximum, and define $\Delta = f(p_{m-1}) - f(p_m)$.

(iii) Calculate $f_3 = f(2p_n - p_0)$ and define $f_1 = f(p_0)$ and $f_2 = f(p_n)$.

(iv) If either $f_3 \geq f_1$ and/or
$$(f_1 - 2f_2 + f_3) \cdot (f_1 - f_2 - \Delta)^2 \geq \frac{1}{2} \Delta (f_1 - f_3)^2$$

use the old directions ξ_1, ξ_2, ..., ξ_n for the next iteration and use p_n for the next p_0, otherwise

(v) defining $\xi = (p_n - p_0)$, calculate λ so that $f(p_n + \lambda\xi)$ is a minimum, use ξ_1, ξ_2, ..., ξ_{m-1}, ξ_{m+1}, ξ_{m+2}, ..., ξ_n, ξ as the directions and $p_n + \lambda\xi$ as the starting point for the next iteration.

Complete theoretical justification of the above algorithm is given in Powell's paper. One effect of step (v) above is that one of the previously determined conjugate directions may be thrown away. In this case the minimum of a quadratic will require more than n iterations. However, Powell shows that this procedure ensures that the rate of convergence will always be reasonable, therefore making the modification valuable. In fact, Powell found it to be essential when minimizing a function of twenty variables, and highly desirable for functions of five variables or more.

Powell suggests a very safe but lengthy convergence criterion. This procedure was not used. Instead, in order to decrease the number of function evaluations required, the iterations were terminated when the results from two successive iterations agreed to within a specified value, ε_p. According to Fletcher [6], only the most difficult functions require the more stringent convergence criterion.

If Powell's method is to be effective, it is essential that an efficient method of finding a minimum along a given direction ξ_i be available. The objective function surface $U(x, r_x)$ is peculiar in that at the boundary of the feasible region the function takes on a

large positive value. Taking this into account the following procedure was developed.

(1) Starting from p_{i-1} use a Fibonacci search along the direction ξ_i to bracket the minimum within the feasible region.

(2) Pass a quadratic through the three points bracketing the minimum and approximate the minimum of the function along the line by the minimum of the quadratic.

(3) Let the quadratic be determined by function values corresponding to design points x_1, x_2, and x_3, and let the minimum of the quadratic be x_{min}. If

$$|x_{min} - x_i| < \epsilon_L, \quad i = 1, 2, 3$$

terminate. Otherwise retain the three points which bracket the minimum and reduce the interval of uncertainty, and repeat step (2).

The above algorithm was used in conjunction with Powell's method and provided convergence comparable to that reported by Powell in his paper.

3.3 Extrapolation

Fiacco and McCormick show that the convergence of SUMT can be accelerated by extrapolation. If p minima of $U(x, r_k)$ have been determined, these may be used to estimate the optimum (the minimum of $V(x)$) and the $(p+1)$st minimum of $U(x, r_k)$.

The estimates of the optimum are shown to converge more quickly to the solution than the sequence of unconstrained minima. In addition the estimate of the (p+1)st minimum can be used as a starting value for the (p+1)st iteration which significantly reduces the amount of computation required to find the optimum.

After p minima have been obtained, estimates of order p-1 can be made. In practice, estimates are seldom made beyond order three due to computer storage requirements and accuracy considerations such as round-off error. The experience of Fiacco and McCormick is that even first and second order approximations of the next $U(x,r_k)$ minimum and the optimum significantly accelerate the convergence.

The extrapolation is based on the fact that the p minima which have been found, x_1, x_2, ..., x_p which correspond to r_1, r_2, ..., r_p may be expanded in terms of the r_k as follows:

$$x_k = \sum_{j=0}^{p-1} a_j (r_k)^j \qquad k = 1, 2, \ldots, p \qquad (3\text{-}8)$$

where a_j are n component vectors. A set of recursion relations based on this expansion leads to the following equations for first and second order estimates of the optimum and the (p+1)st minimum of the U function when $r_{k+1} = r_k/c$, $(c > 1)$.

The first order estimate of the optimum is,

$$x^* = \frac{c x_p - x_{p-1}}{c-1} \qquad (3\text{-}9)$$

and the second order estimate is given by

$$x^* = \frac{c^3 x_p - c^2 x_{p-1} - c x_{p-1} + c x_{p-2}}{(c^2 - 1)(c - 1)} . \tag{3-10}$$

The first order estimate of the (p+1)st minimum is

$$x_{p+1} = \frac{x_p + c x_p - x_{p-1}}{c} \tag{3-11}$$

and the second order estimate may be written

$$x_{p+1} = \frac{c^3 x_p + c^2 x_p - c^2 x_{p-1} - x_{p-2}}{c^3} . \tag{3-12}$$

These equations were used to accelerate convergence when SUMT was applied to the structural optimization problem being considered. The first order estimates were applied after two iterations had been completed and the second order estimates were used thereafter. The estimate of the (p+1)st minimum was used as a starting value for the (p+1)st iteration only if the estimate was a feasible point.

3.4 Parameter Selection and Convergence Criteria

The experience of Fiacco and McCormick indicates that the convergence of SUMT is not greatly effected by either the choice of r_1 or by the choice of the factor C by which r_1 is reduced at each iteration. This was also confirmed by the author's experience when SUMT was applied to the structural optimization problem where values $r_1 = 1$, and $C = 4$ were chosen.

When the differences between components of two successive minima of the $U(x, r_k)$ function were less than a preassigned value, ε_{sx}, the process was terminated. The choice of this preassigned value is not arbitrary. In particular it cannot be less than the value used as a

convergence criterion in the unconstrained minimization. Similarly, the convergence criterion for the one dimensional minimization must be most stringent of all.

CHAPTER IV

CONSTRAINED STEEPEST DESCENT WITH STATE EQUATIONS

4.1 Introduction

The nonlinear programming technique of Chapter III uses the
state equations indirectly to determine the values of the constraints
at a given point in design space. One way of using the state equa-
tions more explicitly in the nonlinear programming problem is to
introduce them as additional constraints on the solution, and then
to use the design variables plus the state variables as the inde-
pendent variables in the problem. Oftentimes, however, this may not
lead to a satisfactory solution, since in many problems the number
of state variables is large compared to the number of design vari-
ables. The dimension of the resulting programming problem becomes
very large. For instance, one of the simplest examples discussed
in Chapter V has six design variables and ten state variables, which
would then lead to a nonlinear programming problem of dimension six-
teen. If the derivatives of the objective function and the con-
straints are cumbersome to compute or are otherwise not available,
the situation is even more acute in that the most successful minimi-
zation techniques, which do not require derivatives, seem to be
limited to problems of about twenty variables [6 & 16].

In this chapter a method is introduced which exploits the state
equations by using them directly in the design process. The general
scheme is to linearize the problem in the neighborhood of a candi-
date design in terms of the design and state variables, and then to
eliminate direct dependence upon the state variables by introducing
the linearized state equations. This results in a new nonlinear pro-
gramming problem to which the Kuhn-Tucker necessary conditions are
applied. As a consequence a step is chosen in design space which
reduces the objective function consistent with the constraints and
simultaneously directs the search for an optimum back towards the
feasible region if any of the constraints have been violated. The
method was developed by Haug [10] who generalized ideas introduced by
Mel'ts [13]. Since the method is not yet available in the literature,
it will be discussed in detail in the following section. Section 4.3
will present the basic algorithm and discuss some of the computational
aspects of the method. Convergence will be considered in Section 4.4
and an expanded algorithm is presented in Section 4.5.

4.2 Description of the Method

In Chapter II it was shown that when the state equations are
included in the formulation, the optimal structural design problem
may be stated as the following mathematical programming problem:

$$\text{minimize} \quad V(x)$$

subject to

$$\phi_i(x) \le 0 \qquad i = 1, 2, \ldots, m \qquad (4\text{-}1)$$

$$h_i(z,x) = 0 \qquad i = 1, 2, \ldots, k$$

$$\psi_i(z) \le 0 \qquad i = 1, 2, \ldots, \ell.$$

With minimal additional effort, the method of constrained steepest descent with state equations (CSDS) can be derived for a more general case which allows the objective function and the ϕ constraints to depend upon the state variables, z, as well as the design variables, x. The more general problem may be stated as follows:

$$\text{minimize} \quad f(z,x)$$

subject to

$$\phi_i(z,x) \le 0 \qquad i = 1, 2, \ldots, m \qquad (4\text{-}2)$$

$$h_i(z,x) = 0 \qquad i = 1, 2, \ldots, k$$

$$\psi_i(z) \le 0 \qquad i = 1, 2, \ldots, \ell.$$

If desired, the ψ constraints could be considered as a subset of the ϕ constraints. If the mathematical form of the two types of constraints is sufficiently similar, this will provide some simplification in the computational algorithm. If the forms of the constraints are considerably different, some computational advantage may be gained by considering them separately. For example, if the ϕ constraints are very simple and the ψ constraints complicated, then it may be advantageous to compute the derivatives of the ϕ constraints

directly but to approximate the derivatives of the ψ constraints. In this section the derivation for the design problem (4-2) is presented

This chapter makes use of matrix calculus notation. A function g of the vector x will be defined as follows:

$$g(x) = \begin{bmatrix} g_1(x) \\ g_2(x) \\ \vdots \\ g_m(x) \end{bmatrix},$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

The derivative of $g(x)$ with respect to x is a matrix,

$$\frac{\partial g}{\partial x} = \begin{bmatrix} \dfrac{\partial g_i}{\partial x_j} \end{bmatrix}$$

of dimension mxn. A small change in x will be denoted by

$$\delta x = \begin{bmatrix} \delta x_1 \\ \delta x_2 \\ \vdots \\ \delta x_n \end{bmatrix},$$

and a first order change in $g(x)$ due to change δx in x is denoted by δg and given by

$$\delta g = \frac{\partial g}{\partial x} \delta x.$$

The CSDS method is derived from the Kuhn-Tucker Necessary Conditions. A proof of this theorem may be found in [5]. It is stated for future reference in this section as follows.

Theorem 4.1 Kuhn-Tucker Necessary Conditions

Let the vectors

$$\begin{bmatrix} \dfrac{\partial \phi_i}{\partial z}^T \\[2ex] \dfrac{\partial \phi_i}{\partial x}^T \end{bmatrix} , \quad i \in \{i: \phi_i(z,x) = 0\}$$

$$\begin{bmatrix} \dfrac{\partial h_i}{\partial z}^T \\[2ex] \dfrac{\partial h_i}{\partial x}^T \end{bmatrix} , \quad i = 1, 2, \ldots, k$$

and

$$\begin{bmatrix} \dfrac{\partial \psi_i}{\partial z}^T \\[2ex] 0 \end{bmatrix} , \quad i \in \{i: \psi_i(z,x) = 0\}$$

be linearly independent at the solution of problem (4-2).
$x = x^*$ and $z = z^*$. Then there exist multipliers $\mu_i \geq 0$,
$i = 1, \ldots, m$, $\nu_i \geq 0$, $i = 1, \ldots, \ell$, and λ_i,
$i = 1, \ldots, k$, such that for

$$H = f(z,x) + \lambda^T h(z,x) + \mu^T \phi + \nu^T \psi, \qquad (4\text{-}3)$$

then $\qquad \partial H / \partial x = 0 \qquad\qquad\qquad\qquad (4\text{-}4)$

$\qquad\qquad \partial H / \partial z = 0 \qquad\qquad\qquad\qquad (4\text{-}5)$

$\qquad\qquad \mu_i = 0 \quad \text{if} \quad \phi_i(z,x) < 0$

and $\qquad \nu_i = 0 \quad \text{if} \quad \psi_i(z) < 0$

at $x = x^*$ and $z = z^*$.

One could attempt to use equations (4-4) and (4-5) of the theorem to construct solutions of problem (4-2), but it is very difficult to determine which of the inequalities are strict equalities at the solution when the number of constraints is large. Instead, the theorem will be used in this section to develop a direct method of solving the optimal design problem.

Let x be an estimate of the solution to problem (4-2) and z be the state variables associated with this solution. The purpose of the method is to determine a small change in x, δx, which will decrease the objective function and satisfy the constraints. To obtain this goal, the problem is first linearized about the estimate to the solution (x,z) where $h(z,x) = 0$. When this is done, the following first order changes in the functions of problems (4-2) result.

$$\delta f = \frac{\partial f}{\partial z} \delta z + \frac{\partial f}{\partial x} \delta x \qquad (4\text{-}6)$$

$$\delta \phi_i = \frac{\partial \phi_i}{\partial z} \delta z + \frac{\partial \phi_i}{\partial x} \delta x \qquad (4\text{-}7)$$

$$\delta h_i = \frac{\partial h_i}{\partial z} \delta z + \frac{\partial h_i}{\partial x} \delta x \qquad (4\text{-}8)$$

$$\delta\psi_i = \frac{\partial\psi_i}{\partial z}\,\delta z \qquad (4\text{-}9)$$

It should be remembered that the partial derivatives in equations (4-6) through (4-9) are evaluated at the estimate (x,z) of the solution. The change in the design variables and the resulting change in the state variables must satisfy $h(z + \delta z, x + \delta x) = 0$. Therefore it is required that $\delta h = 0$. If at the estimate of the solution the inequality constraints are violated, then changes $\delta\phi_i = \Delta\phi_i$, and $\delta\psi_i = \Delta\psi_i$ are requested such that these constraints will be satisfied within the linear formulation of the problem. If $\phi_i(z,x) = 0$ and $\psi_i(z) = 0$ then it is required that $\delta\phi_i = 0$ and $\delta\psi_i = 0$. In addition the accuracy of the linear approximation must be guaranteed by insuring that the step size remains small. Therefore it is required that

$$\delta x^T w \delta x = \xi^2 \qquad (4\text{-}10)$$

where ξ is a small number and w is a positive definite weighting matrix. The linearized version of the problem may then be written as:

$$\text{minimize}\quad \frac{\partial f}{\partial z}\,\delta z + \frac{\partial f}{\partial x}\,\delta x \qquad (4\text{-}11)$$

subject to

$$\frac{\partial\tilde{\phi}}{\partial z}\,\delta z + \frac{\partial\tilde{\phi}}{\partial x}\,\delta x = \Delta\tilde{\phi} \qquad (4\text{-}12)$$

$$\frac{\partial h}{\partial z}\,\delta z + \frac{\partial h}{\partial x}\,\delta x = 0 \qquad (4\text{-}13)$$

$$\frac{\partial\tilde{\psi}}{\partial z}\,\delta z = \Delta\tilde{\psi} \qquad (4\text{-}14)$$

$$\delta x^T w \delta x = \xi^2 \tag{4-15}$$

where

$$\tilde{\phi} = \{\phi_i: \quad \phi_i(z,x) \geq 0\}$$

and
$$\tilde{\psi} = \{\psi_i: \quad \psi_i(z) \geq 0\}.$$

The linearized problem is solved by first eliminating the explicit dependence upon δz and then applying Theorem 4.1. In order to eliminate dependence upon δz, the matrices λ^J, λ^ϕ, and λ^ψ are defined such that

$$\frac{\partial h}{\partial z}^T \lambda^J = \frac{\partial f}{\partial z}^T \tag{4-16}$$

$$\frac{\partial h}{\partial z}^T \lambda^\phi = \frac{\partial \tilde{\phi}}{\partial z}^T \tag{4-17}$$

$$\frac{\partial h}{\partial z}^T \lambda^\psi = \frac{\partial \tilde{\psi}}{\partial z}^T. \tag{4-18}$$

These matrices and equation (4-13) are used to obtain

$$\frac{\partial f}{\partial z} \delta z = \lambda^{J^T} \frac{\partial h}{\partial z} \delta z = -\lambda^{J^T} \frac{\partial h}{\partial x} \delta x \tag{4-19}$$

$$\frac{\partial \tilde{\phi}}{\partial z} \delta z = \lambda^{\phi^T} \frac{\partial h}{\partial z} \delta z = -\lambda^{\phi^T} \frac{\partial h}{\partial x} \delta x \tag{4-20}$$

and
$$\frac{\partial \tilde{\psi}}{\partial z} \delta z = \lambda^{\psi^T} \frac{\partial h}{\partial z} \delta z = -\lambda^{\psi^T} \frac{\partial h}{\partial x} \delta x \tag{4-21}$$

which then allow the linearized problem to be put in the following form:

$$\text{minimize} \quad \lambda^{J^T} \delta x \tag{4-22}$$

subject to

$$\Lambda^{\phi^T} \delta x = \Delta\tilde{\phi} \tag{4-23}$$

$$\Lambda^{\psi^T} \delta x = \Delta\tilde{\psi} \tag{4-24}$$

$$\delta x^T w \delta x = \xi^2. \tag{4-25}$$

The matrices Λ^{J^T}, Λ^{ϕ^T}, and Λ^{ψ^T}, are defined as follows:

$$\Lambda^J = \frac{\partial f}{\partial x}^T - \frac{\partial h}{\partial x}^T \lambda^J \tag{4-26}$$

$$\Lambda^\phi = \frac{\partial\tilde{\phi}}{\partial x}^T - \frac{\partial h}{\partial x}^T \lambda^\phi \tag{4-27}$$

and

$$\Lambda^\psi = - \frac{\partial h}{\partial x}^T \lambda^\psi. \tag{4-28}$$

Theorem 4.1 may now be applied directly to obtain the solution of the linearized problem (4-22) through (4-25). The theorem states that there exist multipliers μ, ν, and ζ, and a function

$$H = \Lambda^{J^T} \delta x + \mu^T(\Lambda^{\phi^T} \delta x - \Delta\tilde{\phi})$$

$$+ \nu^T(\Lambda^{\psi^T} \delta x - \Delta\tilde{\psi}) + \zeta(\delta x^T w \delta x - \xi^2) \tag{4-29}$$

such that

$$\frac{\partial H}{\partial(\delta x)} = \Lambda^{J^T} + \mu^T \Lambda^{\phi^T} + \nu^T \Lambda^{\psi^T} + 2\zeta \delta x^T w = 0. \tag{4-30}$$

This expression can be solved for the required change in x, δx,

$$\delta x = - \frac{1}{2\zeta} w^{-1}(\Lambda^J + \Lambda^\phi \mu + \Lambda^\psi \nu). \tag{4-31}$$

If equation (4-31) is substituted into equations (4-23) and (4-24), one may solve for the multipliers μ and ν. These are given by

$$\mu = -(\Lambda^{\phi^T} w^{-1} \Lambda^\phi)^{-1} \{ 2\zeta \Delta \tilde{\phi} + \Lambda^{\phi^T} w^{-1} \Lambda^J$$

$$- \Lambda^{\phi^T} w^{-1} \Lambda^\psi M_{\psi\psi}^{-1} [2\zeta \Delta \tilde{\psi} - 2\zeta \Lambda^{\psi^T} w^{-1} \Lambda^\phi (\Lambda^{\phi^T} w^{-1} \Lambda^\phi)^{-1} \Delta \tilde{\phi} + M_{\psi J}] \} \quad (4\text{-}32)$$

and

$$\nu = -M_{\psi\psi}^{-1} \{ 2\zeta \Delta \tilde{\psi} - 2\zeta \Lambda^{\psi^T} w^{-1} \Lambda^\phi (\Lambda^{\phi^T} w^{-1} \Lambda^\phi)^{-1} \Delta \tilde{\phi} + M_{\psi J} \}. \quad (4\text{-}33)$$

The matrix $M_{\psi\psi}$ and the vector $M_{\psi J}$ are defined as follows:

$$M_{\psi\psi} = \Lambda^{\psi^T} w^{-1} [I - \Lambda^\phi (\Lambda^{\phi^T} w^{-1} \Lambda^\phi)^{-1} \Lambda^{\phi^T} w^{-1}] \Lambda^\psi \quad (4\text{-}34)$$

$$M_{\psi J} = \Lambda^{\psi^T} w^{-1} [I - \Lambda^\phi (\Lambda^{\phi^T} w^{-1} \Lambda^\phi)^{-1} \Lambda^{\phi^T} w^{-1}] \Lambda^J. \quad (4\text{-}35)$$

If $\tilde{\psi}$ is empty, $M_{\psi\psi}$ is not defined. Similarly, when $\tilde{\phi}$ is empty the
product $\Lambda^{\phi^T} w^{-1} \Lambda^\phi$ is not defined, so that provisions must be made to
properly define equations (4-32) through (4-35) for all cases. This
can be done using the following definitions. If $\tilde{\phi}$ is empty, set
$\Delta\tilde{\phi} = 0$, $\Lambda^\phi = 0$, and $\Lambda^{\phi^T} w^{-1} \Lambda^\phi = I$. If $\tilde{\psi}$ is empty, put $\Delta\tilde{\psi} = 0$, $\Delta\psi = 0$,
and $M_{\psi\psi} = I$. These definitions will reduce equations (4-32) through
(4-35) appropriately. The multipliers (4-32) and (4-33) are then
used to write the required change in the design variables in the fol-
lowing form:

$$\delta x = -\frac{1}{2\zeta} \delta x_1 + \delta x_2,$$

where

$$\delta x_1 = w^{-1} [I - \Lambda^\phi (\Lambda^{\phi^T} w^{-1} \Lambda^\phi)^{-1} \Lambda^{\phi^T} w^{-1}] (\Lambda^J - \Lambda^\psi M_{\psi\psi}^{-1} M_{\psi J}) \quad (4\text{-}36)$$

$$\delta x_2 = w^{-1}[I - \Lambda^{\phi}(\Lambda^{\phi^T}w^{-1}\Lambda^{\phi})^{-1}\Lambda^{\phi^T}w^{-1}]\Lambda^{\psi}M_{\psi\psi}^{-1}$$

$$\cdot[\tilde{\Delta\psi} - \Lambda^{\psi^T}w^{-1}\Lambda^{\phi}(\Lambda^{\phi^T}w^{-1}\Lambda^{\phi})^{-1}\tilde{\Delta\phi}]$$

$$+ w^{-1}\Lambda^{\phi}(\Lambda^{\phi^T}w^{-1}\Lambda^{\phi})^{-1}\tilde{\Delta\phi}. \tag{4-37}$$

The vectors δx_1 and δx_2 have the following properties:

$$\delta x_1^T w \delta x_2 = 0 \tag{4-38}$$

$$\Lambda^{\phi^T}\delta x_2 = \tilde{\Delta\phi} \tag{4-39}$$

$$\Lambda^{\psi^T}\delta x_2 = \tilde{\Delta\psi} \tag{4-40}$$

$$\Lambda^{\phi^T}\delta x_1 = 0 \tag{4-41}$$

$$\Lambda^{\psi^T}\delta x_1 = 0 \tag{4-42}$$

$$\Lambda^{J^T}\delta x_1 \geq 0 \tag{4-43}$$

From these properties and from (4-36) it is seen that δx_1 is a projection of the gradient vector on the planes tangent to the constraints and thus reduces the value of the objective function consistent with the constraints. The vector δx_2 is a correction vector which directs the solution towards the feasible region. Furthermore these two vectors are orthogonal with respect to the weighting matrix w and are therefore in that sense independent.

The properties of δx_1 and δx_2, and equation (4-25), can be used to determine the parameter ζ.

$$\delta x^T w \delta x = \frac{1}{4 \zeta^2} \delta x_1^T w \delta x_1 + \delta x_2^T w \delta x_2 = \xi^2 \qquad (4\text{-}44)$$

From this expression it is clear that ζ is not arbitrary. In fact, it depends on ξ which from $(4\text{-}44)$ must satisfy

$$\xi^2 \geq \delta x_2^T w \delta x_2. \qquad (4\text{-}45)$$

In addition, if there are no constraints or state variables, the expression for δx reduces to

$$\delta x = -\frac{1}{2\zeta} \Lambda^J,$$

and the move to a new approximation of the minimum is in the direction of steepest descent. Therefore, it is required that $\zeta > 0$. One pro-cedure that can be applied to determine the size of step made in the direction of δx is to choose ξ and then use $(4\text{-}44)$ to compute ζ. Alternatively, one can use the following expression,

$$\delta x = -\eta \delta x_1 + \delta x_2$$

and choose $\eta > 0$ and small. The problem of choosing the step size will be considered further in the next section along with other com-putational considerations.

The constrained steepest descent technique has been developed for problem $(4\text{-}2)$. The optimal structural design problem being con-sidered in this research has been cast in the form of problem $(4\text{-}1)$ which is included in problem $(4\text{-}2)$ as a special case. All that is required is to note that, since the objective function and the ϕ

constraints are not functions of the state variables, $\lambda^J = 0$ and $\lambda^\phi = 0$ from equations (4-16) and (4-17).

This simplifies the expressions for Λ^J and Λ^ϕ — equations (4-26) and (4-27). The remaining expressions are not changed.

4.3 Computational Considerations

The general procedure for determining the optimum of problem (4-1) or (4-2) is outlined below.

(1) Make an estimate of the optimal design vector, $x^{(0)}$.

(2) Solve for the state variables, z, corresponding to the design vector $x^{(j)}$ of the current iteration.

(3) Solve (4-16), (4-17), and (4-18) for λ^J, λ^ϕ, and λ^ψ.

(4) Determine Λ^J, Λ^ϕ, and Λ^ψ, from equations (4-26), (4-27), and (4-28).

(5) Choose $\Delta\tilde{\phi}$ and $\Delta\tilde{\psi}$ and compute $M_{\psi\psi}$ and $M_{\psi J}$ from equations (4-34) and (4-35).

(6) Compute δx_1 and δx_2 from (4-36) and (4-37).

(7) Choose $\eta > 0$ and compute
$$x^{j+1} = x^j - \eta\delta x_1 + \delta x_2.$$

(8) Check for convergence and terminate or go to (2).

There are several points in the above algorithm that require further comment. First of all, in the course of computations two matrices, $(\Lambda^{\phi^T} w^{-1}\Lambda^\phi)$ and $M_{\psi\psi}$, have to be inverted and therefore must be non-singular. Secondly, the quantities $\Delta\tilde{\phi}$ and $\Delta\tilde{\psi}$ must be chosen. A

third consideration is the choice of the appropriate step size as determined by the choice of η, and a fourth concern is related to the tendency of the method to keep a constraint satisfied once it becomes identically zero.

The two matrices of concern are positive semi-definite. In the computational procedure they are assumed to be positive definite. Experience with the algorithm has shown that this assumption is almost always valid. Certain pathological cases can arise, however, when this is not true. An examination of equation (4-34) shows that $M_{\psi\psi}$ will be singular when the columns of Λ^{ψ} are linearly dependent. This follows from the fact that the rank of a product of two matrices cannot exceed the rank of either factor. Similarly, linear dependence between columns of Λ^{ψ} will make the product $(\Lambda^{\phi^T} w^{-1} \Lambda^{\phi})$ singular. Examination of equations (4-17), (4-18), (4-27), and (4-28) shows that the matrices Λ^{ϕ} and Λ^{ψ} are closely related to the gradients of the ϕ and ψ constraints. For example, when the ϕ constraints are not dependent upon the state variables (as in the structural problem considered), then the columns of the matrix Λ^{ϕ} are just the gradient vectors of the violated or satisfied ϕ constraints. If these gradient vectors are linearly dependent then it follows directly that $(\Lambda^{\phi^T} w^{-1} \Lambda^{\phi})$ is singular. Furthermore, the assumptions in the Kuhn-Tucker necessary conditions are not satisfied. The matrix $M_{\psi\psi}$ can also become singular if Λ^{ϕ} is a square matrix. This will happen when the number of ϕ constraints violated is equal to the number of

design variables. The direct consequence is that the product $\Lambda^\phi(\Lambda^{\phi^T} w^{-1} \Lambda^\phi)^{-1} \Lambda^{\phi^T} w^{-1}$ becomes the identity matrix making $M_{\psi\psi} = 0$.

Other such cases could probably be identified. The author's experience indicates that these are the exception, rather than the rule, and resulted from the introduction of redundant constraints or a poor choice of η in determining step size. In all cases the cause of the singularity was readily identified and corrected without altering the basic algorithm.

Equation (4-37), which is used to compute δx_2, requires the quantities $\Delta\tilde{\phi}$ and $\Delta\tilde{\psi}$. The vector δx_2 is the correction component of the vector δx which directs the search for the optimum back towards the feasible region. The magnitude of this correction depends upon the values assigned to the vectors $\Delta\tilde{\phi}$ and $\Delta\tilde{\psi}$. In the structural design problem the values $\Delta\tilde{\phi} = -\tilde{\phi}$, and $\Delta\tilde{\psi} = -\tilde{\psi}$, were assigned. This choice is simple and effective.

The value of η controls both the direction and magnitude of the change in the design variable vector δx. Several methods of choosing η were tried. The most successful of these was to specify a maximum value of η and to use $\eta = \eta_{max}$ unless the constraints were violated beyond certain preassigned limits. For example, for a maximum allowable stress of 40,000 psi, $\eta = \eta_{max}$ was used unless the stresses obtained using this value exceeded 44,000 psi. In other words a ten per cent violation of the constraints was allowed. This sped up convergence by allowing bigger steps to be taken without excessively

violating the constraints. If an unacceptable violation of the constraints occurred, the current value f_η was reduced by ten per cent and the constraints were checked again. This process was continued until an acceptable value of η was obtained or until η became zero. If this occurred, the value of η was set to 10^{-6} and computations were allowed to proceed. A zero value of η is not allowed since at other points in the algorithm division by η is required.

When an estimate of the solution is such that some of the constraints are exactly satisfied, the method demands that they continue to be satisfied. It may be advantageous if the search for the minimum is allowed to leave such a constraint. Theorem 4.1 states that the multipliers, μ and ν, which exist at a solution, must be positive or zero but that they cannot be negative. Therefore, at each successive iteration of the optimization process the multiplier vectors μ and ν are computed. If either of these has components which are negative, the constraints corresponding to these components are removed from the appropriate constraint set $\tilde{\phi}$ or $\tilde{\psi}$. This procedure proved effective when the method was applied to the optimal structural design problem.

4.4 Convergence Criterion

A convergence criterion must be established for Step (8) in the algorithm of Section 4.3. It will be shown that, if the sequence of solutions of the linearized problem converges to a local solution of the nonlinear problem, then $\delta x_1 = 0$. The result is obtained by the

direct application of the Kuhn-Tucker necessary conditions to both the nonlinear and the linearized problems and comparison of the resulting equations.

Theorem 4.1 states that at the solution of the nonlinear problem (4-2) there exist multipliers

$$\mu_i \geq 0, \qquad i = 1, \ldots, m$$

$$\nu_i \geq 0, \qquad i = 1, \ldots, \ell$$

and
$$\lambda_i, \qquad i = 1, \ldots, k$$

such that for

$$H = f(z,x) + \lambda^T h(z,x) + \mu^T \phi(z,x) + \nu^T \psi(z)$$

then

$$\frac{\partial H}{\partial x} = \frac{\partial f}{\partial x} + \lambda^T \frac{\partial h}{\partial x} + \mu^T \frac{\partial \phi}{\partial x} = 0 \qquad (4\text{-}46)$$

$$\frac{\partial H}{\partial z} = \frac{\partial f}{\partial z} + \lambda^T \frac{\partial h}{\partial z} + \mu^T \frac{\partial \phi}{\partial z} + \nu^T \frac{\partial \psi}{\partial z} = 0 \qquad (4\text{-}47)$$

$$\mu_i = 0 \quad \text{if } \phi_i(z,x) < 0$$

and
$$\nu_i = 0 \quad \text{if } \psi_i(z) < 0.$$

Define $\tilde{\phi}$ and $\tilde{\psi}$ such that

$$\tilde{\phi} = \{\phi_i: \quad \phi_i(z,x) \geq 0\} \qquad (4\text{-}48)$$

$$\tilde{\psi} = \{\psi_i: \quad \psi_i(z) \geq 0\} \qquad (4\text{-}49)$$

and $\tilde{\mu}$ and $\tilde{\nu}$ which contain only the components of μ and ν respectively, corresponding to the constraints in $\tilde{\phi}$ and $\tilde{\psi}$. Using this information, equations (4-46) and (4-47) can be written in the following form,

$$\frac{\partial f}{\partial x} + \lambda^T \frac{\partial h}{\partial x} + \tilde{\mu}^T \frac{\partial \tilde{\phi}}{\partial x} = 0 \qquad (4\text{-}50)$$

$$\frac{\partial f}{\partial z} + \lambda^T \frac{\partial h}{\partial z} + \tilde{\mu}^T \frac{\partial \tilde{\phi}}{\partial z} + \tilde{\nu}^T \frac{\partial \tilde{\psi}}{\partial z} = 0. \qquad (4\text{-}51)$$

Using equations (4-16) through (4-18), equation (4-51) may be written

$$\Lambda^{J^T} \frac{\partial h}{\partial z} + \lambda^T \frac{\partial h}{\partial z} + \tilde{\mu}\lambda^{\phi^T} \frac{\partial h}{\partial z} + \tilde{\nu}^T \lambda^{\psi^T} \frac{\partial h}{\partial z} = 0. \qquad (4\text{-}52)$$

The matrix $\frac{\partial h}{\partial z}$ is nonsingular, therefore

$$\lambda^J + \lambda + \lambda^\phi \tilde{\mu} + \lambda^\psi \tilde{\nu} = 0. \qquad (4\text{-}53)$$

Equation (4-53) may be used to eliminate λ from equation (4-52) to obtain

$$\frac{\partial f}{\partial x} - (\Lambda^{J^T} + \tilde{\mu}\lambda^{\phi^T} + \tilde{\nu}^T \lambda^{\psi^T}) \frac{\partial h}{\partial x} + \tilde{\mu}^T \frac{\partial \tilde{\phi}}{\partial x} = 0, \qquad (4\text{-}54)$$

Collecting coefficients on $\tilde{\mu}$ and $\tilde{\nu}$, and using equations (4-26) through (4-28) yields,

$$\Lambda^\psi \tilde{\mu} = -(\Lambda^J + \Lambda^\psi \tilde{\nu}). \qquad (4\text{-}55)$$

If equation (4-55) is premultiplied by $\Lambda^{\phi^T} w^{-1}$, the following expression for $\tilde{\mu}$ is obtained

$$\tilde{\mu} = -(\Lambda^{\phi^T} w^{-1} \Lambda^\phi)^{-1} \Lambda^{\phi^T} w^{-1} (\Lambda^J + \Lambda^\psi \tilde{\nu}) \qquad (4\text{-}56)$$

It should be noted that it has been assumed that $(\Lambda^{\phi^T} w^{-1} \Lambda^\phi)$ is non-singular. If equation (4-56) is substituted into (4-55),

$$\left[I - \Lambda^\phi (\Lambda^{\phi^T} w^{-1} \Lambda^\phi)^{-1} \Lambda^{\phi^T} w^{-1} \right] \left[\Lambda^J + \Lambda^\psi \tilde{\nu} \right] = 0 \qquad (4\text{-}57)$$

is obtained. Premultiplying by $\Lambda^{\phi^T} w^{-1}$ and using equations (4-34) and (4-35), equation (4-57) becomes

$$M_{\psi J} + M_{\psi\psi}\tilde{\nu} = 0$$

or

$$\tilde{\nu} = -M_{\psi\psi}^{-1}M_{\psi J}. \qquad (4\text{-}58)$$

As a result equation (4-57) may be written

$$[I - \Lambda^{\phi}(\Lambda^{\phi^T}W^{-1}\Lambda^{\phi})^{-1}\Lambda^{\phi^T}W^{-1}][\Lambda^{J} - \Lambda^{\psi}M_{\psi\psi}^{-1}M_{\psi J}] = 0. \qquad (4\text{-}59)$$

Therefore by equation (4-36), at the solution of problem (4-2), it is necessary that $\delta x_1 = 0$. Furthermore in the limit $\tilde{\phi} = 0$ and $\tilde{\psi} = 0$ so that $\Delta\tilde{\phi} = 0$ and $\Delta\tilde{\psi} = 0$ and by equation (4-37), $\delta x_2 = 0$. Since

$$\delta x = -\eta\delta x_1 + \delta x_2, \qquad (4\text{-}60)$$

δx must also approach zero if the procedure converges. Satisfaction of this criterion is used in the algorithm given in the next section.

4.5 CSDS Algorithm

The following algorithm is an expansion of the procedure defined in Section 4.3 and incorporates the additional features discussed in Sections 4.3 and 4.4.

(1) Estimate the optimal design vector, $x^{(0)}$.

(2) Determine the state variables z corresponding to the design variables of the current iteration, $x^{(j)}$.

(3) Solve (4-16), (4-17), and (4-18) for λ^J, λ^ϕ, and λ^ψ.

(4) Determine $\tilde{\phi}$ and $\bar{\psi}$.

(5) Determine Λ^J, Λ^ϕ, and Λ^ψ from equations (4-26), (4-27), and (4-28).

(6) Choose $\Delta\tilde{\phi} = -\tilde{\phi}_i$ and $\Delta\bar{\psi}_i = -\bar{\psi}_i$.

(7) Compute $M_{\psi\psi}$ and $M_{\psi J}$ from equations (4-34) and (4-35).

(8) Determine the multipliers μ and ν from (4-32) and (4-33).

(9) If any of the components of μ and ν are negative, remove the corresponding constraints from the sets $\tilde{\phi}$ and $\tilde{\psi}$ and return to Step (5). If all components of μ and ν are non-negative, continue.

(10) Compute δx_1 and δx_2 from (4-36) and (4-37).

(11) Choose $\eta = \eta_{max}$.

(12) If $\eta \leq 0$, set $\eta = 10^{-6}$ and go to Step (15).

(13) Compute

$$x^{j+1} = x^j - \eta\delta x_1 + \delta x_2.$$

(14) Compute the values of the constraints for x^{j+1}. If the constraints have been violated excessively, reduce n and return to Step (11), otherwise continue.

(15) If $\delta x_i < \epsilon_x$, $i = 1, \ldots, n$ terminate. Otherwise return to Step (2).

The application of this algorithm to the optimal structural design problem, and the ensuing results, are discussed in Chapter V.

CHAPTER V

APPLICATIONS

5.1 General Discussion

The optimization methods presented in Chapter III and Chapter IV
were applied to the design of several two and three member frames sub-
jected to a variety of loads. In order to aid discussion the frame
members and loads are numbered as shown in Figure 5.1 and Figure 5.2.
All dimensions and limits on dimensions are given in inches, the loads
are in pounds and the stresses are in pounds per square inch. In
Section 5.2 and 5.3 a general description of the frames to be opti-
mized is given. Sections 5.4 and 5.5 present a brief discussion of
the programs used and some aspects of their application. The result-
ing designs are presented and discussed in Sections 5.6 and 5.7.

5.2 The Two Member Frame

The two member frames which were optimized are shown in Tables
5.1 through 5.10. In each case there were six design variables to
be determined, ten state variables, and seventeen constraints to be
satisfied. The design variables are the wall thickness, t, width, b,
and height, h, (see Figure 2.2) of each member. These variables must
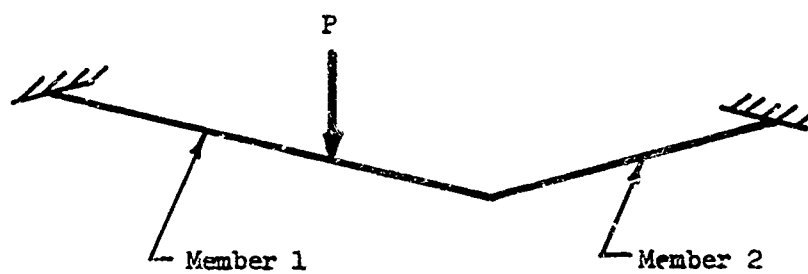satisfy the following inequalities for the ith member:
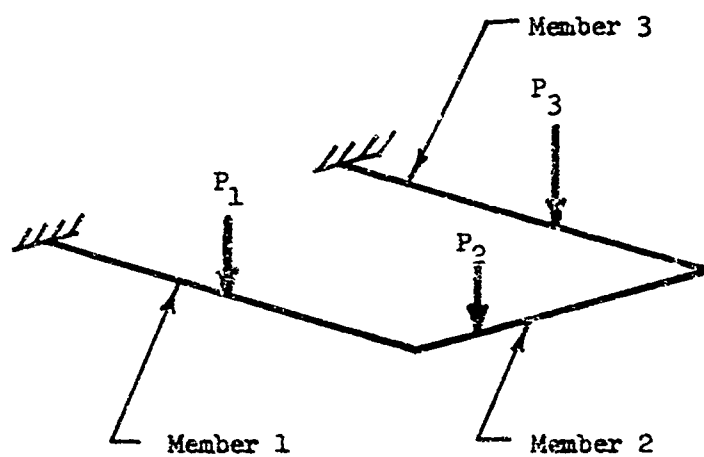
FIGURE 5.1 Numbering the Two Member Frame

FIGURE 5.2 Numbering the Three Member Frame

$$0.1 \leq t_i \leq 1.0$$
$$2.5 \leq b_i \leq 10.0 \qquad (5-1)$$
$$2.5 \leq h_i \leq 10.0.$$

The state variables are the three displacement components of the joint, the torsional stress in each member, and the bending stresses at the five critical points of the structure. The critical points are the ends of each member and under the applied load. The stresses must satisfy the following inequality at the ith critical point:

$$\sqrt{\sigma_i^2 + 3\tau_i^2} - 40,000 \leq 0 \qquad (5-2)$$

where σ_i is the normal stress due to bending and τ_i is the shear stress due to torsion at the point.

The sequential unconstrained minimization technique and the method of constrained steepest descent with state equations were both applied to the two member frames. The comparative effectiveness of these two approaches is discussed in Section 5.6.

5.3 The Three Member Frame

The three member frames optimized are shown in Tables 5.11 through 5.17. For each frame nine design variables were chosen subject to eighteen design variable constraints. There are eighteen state variables and twelve state variable constraints. The design variables are the same as those for the two member frame. The state variables consist of six joint displacements and twelve stresses. The stresses are the torsional stress in each member plus three

critical bending stresses in each member occurring at the ends or under the applied load. For the frames in Tables 5.11 through 5.15, the stresses at the ith critical point were required to satisfy the condition (5-2), and the design variables were required to satisfy (5-1).

The dimensions and loads shown in Tables 5.16 and 5.17 were chosen to approximate a frame that the author encountered in an industrial application. In both cases the stresses at a critical point are required to satisfy the following condition:

$$\sqrt{\sigma_i^2 + 3\tau_i^2} - 20,000 \leq 0. \tag{5-3}$$

The design variables for the frame in Table 5.15 satisfy

$$.109 \leq t_i \leq 1.0$$
$$2.0 \leq b_i \leq 10.0 \tag{5-4}$$
$$6.0 \leq h_i \leq 10.0 \quad \text{for } i = 1, 3,$$

and

$$.25 \leq t_2 \leq 1.0$$
$$3.0 \leq b_2 \leq 10.0 \tag{5-5}$$
$$3.0 \leq h_2 \leq 10.0.$$

The design variable constraints which the frame in Table 5.16 satisfied are,

$$.109 \leq t_i \leq 1.0$$
$$2.0 \leq b_i \leq 5.0 \tag{5-6}$$
$$6.0 \leq h_i \leq 12.0$$

for members 1 and 3, and (5-5) for member 2.

Because of its superior performance on the two member frames, only the method of constrained steepest descent with state equations was used in the optimum design of the three member frames.

5.4 The SUMT Program

The sequential unconstrained minimization technique was programmed using the algorithms of Sections 3.1 and 3.2. Acceleration by extrapolation was included and proved to be effective both in decreasing the number of function evaluations and in giving better convergence to the optimum, but only when the calculations were done in double precision. Fiacco and McCormick alude to this when they discuss the significant effect of round-off error in the extrapolation procedures.

Certain safeguards must be introduced into the program to insure that the search for a minimum always takes place within the feasible region as required by the development in Chapter III. In the Fibonacci search increasingly larger steps are taken until the minimum is bracketed. When the minimum is near the boundary of the feasible region, it is possible for a step to violate a nearby constraint. Therefore any time a new step is taken in the optimization process a check must be made to see if a constraint has been violated. If violation occurs, then appropriate measures must be taken to return the search to the feasible region. These procedures are ad hoc and significantly reduce the efficiency of the unconstrained minimization.

Three convergence criteria, ε_{sx}, ε_{p}, and ε_{L}, must be chosen for SUMT. Of these ε_{L} should have the smallest value, since the greatest accuracy is required in the minimization of a function along a line. ε_{sx} cannot be less than the values of either ε_{p} or ε_{L}. The choice of these parameters significantly affects the convergence of the method and in almost all cases experimentation was required before a satisfactory set was obtained.

5.5 CSDS Program

The constrained steepest descent program was written using the algorithm of Section 4.5. All computations are done in double precision and the program is written so that the required derivatives can be computed exactly or from finite difference approximations. Derivatives of the state equations, constraint functions, and objective functions are required by the algorithm. The derivatives of the state equations with respect to the design variables are the only derivatives approximated since the remaining functions are simple and the derivatives can be computed exactly without undue effort. Each frame shown in the tables was optimized using both exact and approximate derivatives. A simple forward difference scheme was used initially to obtain the approximate derivatives. Some difficulty was experienced but results were improved when the interval of the approximation was reduced. Still, for some three member frames, the method did not converge properly when the derivatives were approximated.

This problem was eliminated at the expense of more function evaluations by using a central difference approximation.

Two parameters, ε_x and η_{max}, must be chosen when the program is used. η_{max} is the maximum step size allowed and ε_x is the convergence criterion. A value of $\varepsilon_x = .1 \times 10^{-4}$ was chosen. This is somewhat more stringent than the value of ε_{sx} used is SUMT; but if the method converges at all, it has no difficulty satisfying this criterion. Some minimal experimenting may be necessary to choose η_{max}. For large values of η_{max}, fewer iterations are required if the method converges, but sometimes situations may be created which the algorithm cannot correct. These are immediately apparent and easily eliminated by reducing the value of η_{max}.

5.6 Application to Two Member Frames

Three different design problems are shown in Tables 5.1 through 5.10. These designs were optimized using both the sequential unconstrained minimization technique and the method of constrained steepest descent with state equations. The frame shown in Tables 5.1 through 5.4 has members of different lengths with the load applied at the joint. The computation time for the solution using SUMT (Table 5.1) is considerably greater than the time required using CSDS (Tables 5.2 through 5.3), even though the convergence criterion ε_{sx} for SUMT is considerably less restrictive than the criterion ε_x used in CSDS. In SUMT the state of the structure is computed for each function evaluation. In CSDS the state is computed about ten times per iteration.

Using this value the number of functional evaluations required for the design of Table 5.3 is about 140 compared to the 6,633 evaluations required for SUMT. The results of Tables 5.1 through 5.3 show that different starting values converge to the same minimum. However, when the starting values of Table 5.1 were used in the CSDS method, another local minimum was obtained. This is not surprising in view of the nonconvex nature of the programming problem.

Another frame was optimized using SUMT (Tables 5.5 through 5.6) and CSDS (Table 5.7). The large difference in computing time for the two methods is again apparent. Starting from the same values, the two methods converged to two different local minima. Several attempts were made to start in the neighborhood of the minimum determined by SUMT using CSDS, but the latter technique always converged to the symmetric results of Table 5.7. An additional trial (Table 5.6) was made using SUMT with a starting value in the neighborhood of the symmetric solution. The results are close to those of Table 5.7. The volume is smallest for the symmetrical design.

The third two member frame that was optimized is shown in Tables 5.8 through 5.10. Two trials using SUMT and one using CSDS were made. The resulting volumes do not differ greatly, but the design variables show significant differences. About 840 function evaluations (assuming ten per iteration) are required when CSDS is used compared to 11,610 required by SUMT (Table 5.9). The amount of computing time required shows a similar advantage for CSDS.

The results show that the method of constrained steepest descent with state equations has significant advantages over the sequential unconstrained minimization technique. Both the computation time and the number of times that the state of the structure must be evaluated are considerably less than those required for SUMT. Furthermore, the same results are obtained with or without the exact calculation of derivatives.
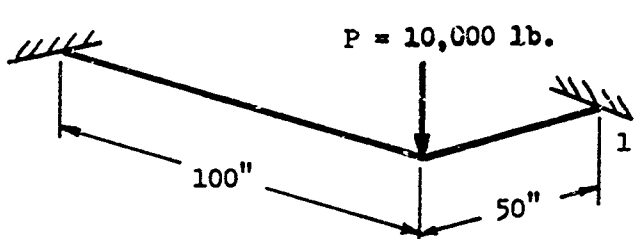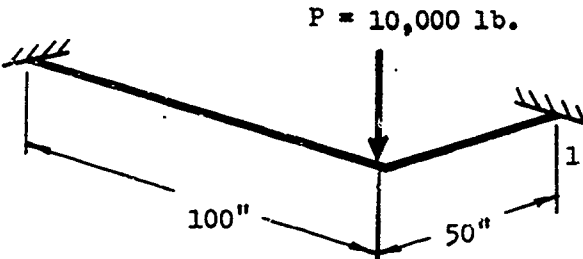
$\sigma_f = 39,945$ psi. at 1

		Initial Values, in.	Optimum Values, in.
$\varepsilon_{sx} = .1 \times 10^{-2}$	t_1	0.9	0.1010
$\varepsilon_x = .1 \times 10^{-4}$	b_1	9.0	2.5070
$\varepsilon_1 = .1 \times 10^{-5}$	h_1	9.0	2.5070
	t_2	0.9	0.1012
	b_2	9.0	9.4234
	h_2	9.0	9.9827
Volume, in^3		4,374.00	291.59
Number of iterations			6
Computing time, sec.			370
Number of function evaluations			8,633

TABLE 5.1 Optimum Design of a Two Member Frame Using SUMT
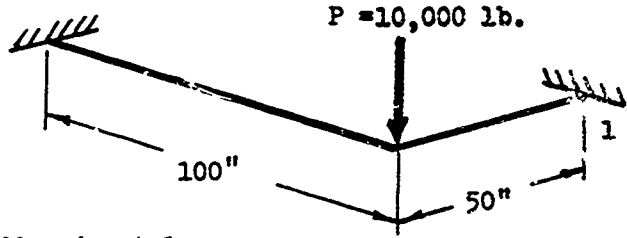
σ_f = 40,000 psi. at 1

		Initial Values, in.	Optimum Values, in. Obtained Using	
			Exact Derivatives	Approximate Derivatives
η_{max} = .1 ϵ_x = .1x10^{-4}	t_1	0.10	0.10000	0.10000
	b_1	2.50	2.50000	2.50000
	h_1	2.50	2.50000	2.50000
	t_2	0.11	0.10000	0.10000
	b_2	10.00	9.52614	0.52614
	h_2	10.00	10.00000	10.00000
Volume, in.^{3}		313.58	289.26	289.26
Number of iterations			9	9
Computing time, sec.			20	20

TABLE 5.2   Optimum Design of a Two Member Frame Using CSDS
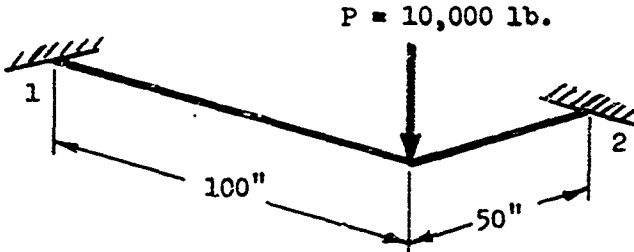
		Initial Values, in.	Optimum Values, in. Obtained Using	
			Exact Derivatives	Approximate Derivatives
$n_{max} = .1$	t_1	0.11741	0.10000	0.10000
$\epsilon_x = .1 \times 10^{-4}$	b_1	2.5246	2.50000	2.50000
	h_1	2.5217	2.50000	2.50000
	t_2	0.22061	0.10000	0.10000
	b_2	3.2585	9.52614	9.52614
	h_2	3.2544	10.00000	10.00000
Volume, in^3		246.93	289.26	289.26
Number of iterations			14	14
Computing time, sec.			14	14

TABLE 5.3   Optimum Design of a Two Member Frame Using CSDS

P = 10,000 lb.

1

100"

50"

2
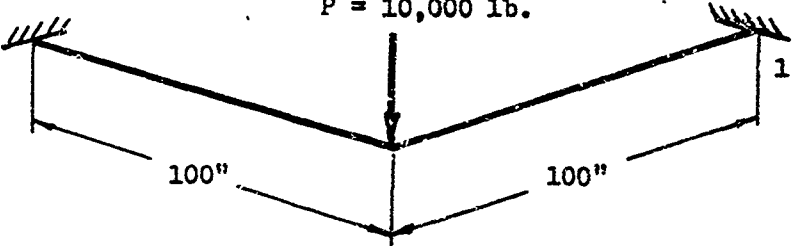
σ_f = 40,000 psi. at 1,2

		Initial Values	Optimum Values, in. Obtained Using	
			Exact Derivatives	Approximate Derivatives
η_{max} = .1 ϵ_x = .1x10^{-4}	t_1	0.9	0.11373	0.11373
	b_1	9.0	10.00000	10.00000
	h_1	9.0	10.00000	10.00000
	t_2	0.9	0.23599	0.23598
	b_2	9.0	10.00000	10.00000
	h_2	9.0	2.50000	2.50000
Volume, in.^{3}		4,374.00	733.59	733.59
Number of iterations			19	19
Computing time, sec.			18	18

TABLE 5.4   Optimum Design of a Two Member Frame Using CSDS

		Initial Values, in.	Optimum Values, in.
$\epsilon_{sx} = .1 \times 10^{-2}$	t_1	0.9	0.1001
$\epsilon_x = .1 \times 10^{-4}$	b_1	9.0	2.5004
$\epsilon_1 = .1 \times 10^{-4}$	h_1	9.0	2.5005
	t_2	0.9	0.1971
	b_2	9.0	9.9969
	h_2	9.0	9.9981
Volume, in.^{3}		5,832.0	868.87
Number of iterations			10
Computing time, sec.			190
Number of function evaluations			5,598

TABLE 5.5 Optimum Design of a Two Member Frame Using SUMT
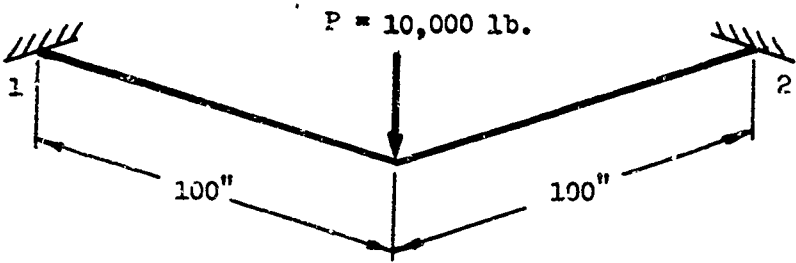
$P = 10,000$ lb.

1 2

$100''$ $100''$

$\sigma_f = 39,998$ psi. at 1,2

		Initial Values, in.	Optimum Values, in.
$\varepsilon_{sx} = .1 \times 10^{-2}$	t_1	0.15	0.1007
$\varepsilon_x = .1 \times 10^{-3}$	b_1	9.90	9.9938
$\varepsilon_1 = .1 \times 10^{-4}$	h_1	9.90	9.9978
	t_2	0.15	0.1007
	b_2	9.90	8.9359
	h_2	9.90	9.9976
Volume, in^3		1,170.00	775.89
Number of iterations			11
Computing time, sec.			353
Number of function evaluations			10,414

TABLE 5.6 Optimum Design of a Two Member Frame Using CSDS
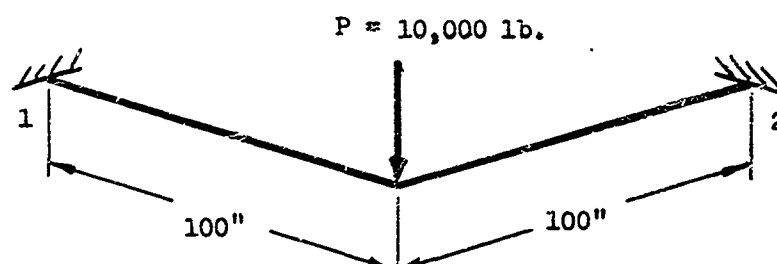
$$P = 10,000 \text{ lb.}$$

100"   100"

$\sigma_f = 40,000$ psi. at 1,2

		Initial Values, in.	Optimum Values, in. Obtained Using	
			Exact Derivatives	Approximate Derivatives
$\eta_{max} = .1$	t_1	0.9	0.10000	0.10000
$\varepsilon_x = .1 \times 10^{-4}$	b_1	9.0	9.54738	9.54738
	h_1	9.0	10.00000	10.00000
	t_2	0.9	0.10000	0.10000
	b_2	9.0	9.54738	9.54738
	h_2	9.0	10.00000	10.00000
Volume, in^3		5,832.00	773.89	773.89
Number of iterations			15	15
Computing time, sec.			22	22

TABLE 5.7 Optimum Design of a Two Member Frame Using CSDS
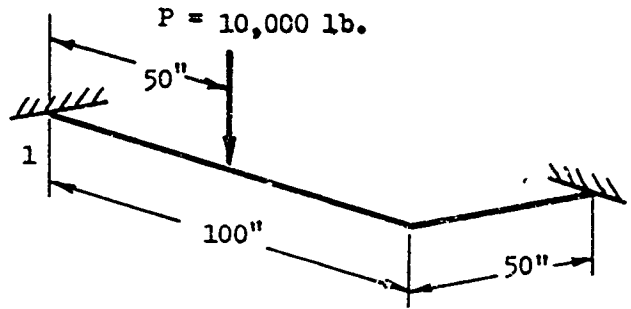
			Initial Values, in.	Optimum Values, in.
$\varepsilon_{sx} = .1 \times 10^{-1}$		t_1	0.2	0.100
$\varepsilon_x = .1 \times 10^{-3}$		b_1	5.0	4.319
$\varepsilon_1 = .1 \times 10^{-4}$		h_1	9.0	8.253
		t_2	0.2	0.100
		b_2	3.0	2.502
		h_2	9.0	9.075
Volume, in³			776.00	362.07
Number of iterations				9
Computing time, sec.				310
Number of function evaluations				11,251

TABLE 5.8 Optimum Design of a Two Member Frame Using SUMT
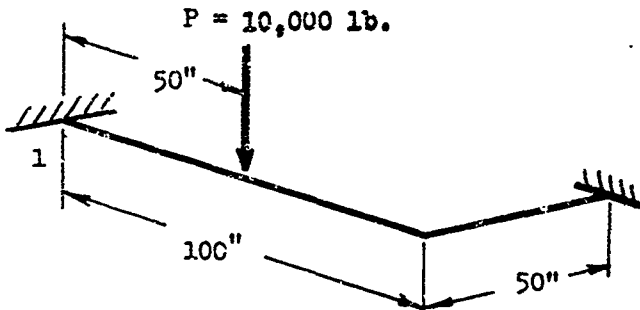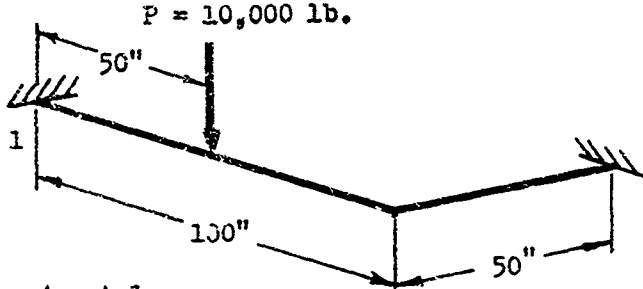
TABLE 5.9 Optimum Design of a Two Member Frame Using SUMT

P = 10,000 lb.

50"

1

100"

50"

σ_f = 39,994 psi. at 1

		Initial Values, in.	Optimum Values, in.
$\epsilon_{sx} = .1\text{x}10^{-1}$	t_1	0.9	0.100
$\epsilon_x = .1\text{x}10^{-3}$	b_1	9.0	4.196
$\epsilon_1 = .1\text{x}10^{-4}$	h_1	9.0	8.280
	t_2	0.9	0.100
	b_2	9.0	2.502
	h_2	9.0	9.381
Volume, in³		5,832.0	363.18
Number of iterations			9
Computing time, sec.			395
Number of function evaluations			11,610

TABLE 5.9 Optimum Design of a Two Member Frame Using SUMT

		Initial Values, in.	Optimum Values, in. Obtained Using	
			Exact Derivatives	Approximate Derivatives
$\eta_{max} = .1$	t_1	0.9	0.10000	0.10000
$\varepsilon_x = .1 \times 10^{-4}$	b_1	9.0	4.40827	4.40827
	h_1	9.0	8.58807	8.58807
	t_2	0.9	0.10000	0.10000
	b_2	9.0	2.50000	2.50000
	h_2	9.0	8.07038	8.07038
Volume, in.^{3}		4,374.00	359.63	359.63
Number of iterations			84	84
Computing time, sec.			72	72

TABLE 5.10   Optimum Design of a Two Member Frame Using CSDS

5.7 Application to Three Member Frames

The results of Section 5.6 show the advantages of the method of constrained steepest descent with state equations over the sequential unconstrained minimization technique when optimization without computing exact derivatives is required. Furthermore, as the number of design variables is increased, SUMT will become increasingly more difficult to apply because of the nature of the algorithms used to determine the unconstrained minima. Therefore, the optimization of the three member frames was carried out by using only CSDS.
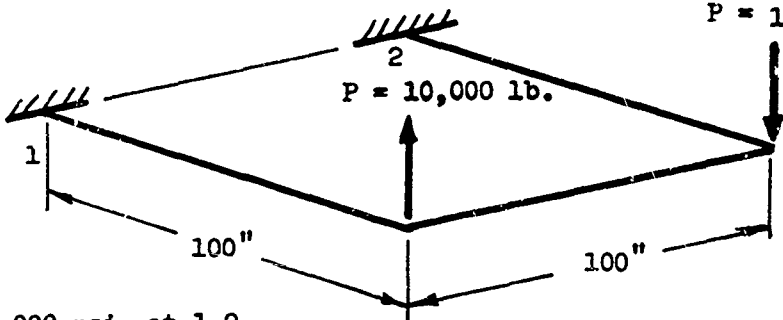
A variety of symmetrical and unsymmetrical load sets were applied to several different three member frames. The minimum weight designs obtained are shown in Tables 5.11 through 5.17. The agreement between designs obtained using exact and approximate derivatives is particularly noteworthy. This was improved during the course of the applications by introducing a central difference formula to approximate the derivatives of the state equations. The improvement that resulted may be seen by comparing the results of Tables 5.13 and 5.14. The designs of Tables 5.15 through 5.17 also show the value of this modification, since the results for these frames using exact and approximate derivatives agree exactly.

The application in Table 5.12 illustrates another very useful feature of CSDS. In this case the method did not converge within the maximum allowable iterations, but the value of δx had decreased and was of the order 10^{-2}. Thus, the necessary condition that $\delta x = 0$

(Section 4.4) is approximately satisfied, indicating that the results are probably in the neighborhood of a minimum weight design. Consequently, further iterations are not absolutely necessary. Another feature of CSDS shown by the results is that constraints may be satisfied exactly. In SUMT (see Section 5.6) constraints are only approximately satisfied.

An unsymmetrical frame (Table 5.15) required several attempts to obtain the design shown. This is not uncommon in that some experimentation is almost always required to choose ϵ_x and n_{max} properly. Furthermore an unfavorable starting point may be inadvertently chosen.
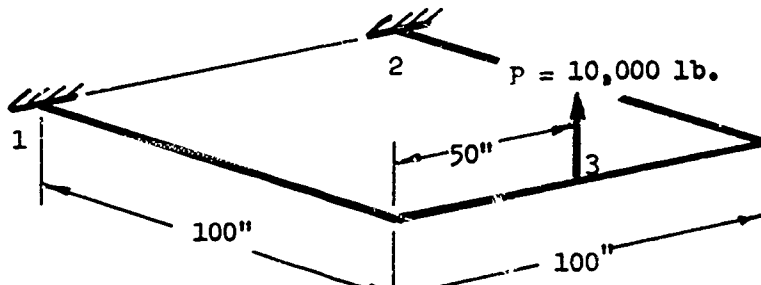
The designs presented in Tables 5.16 and 5.17 illustrate the dependency of the results both on the configuration of the basic frame and the assignment of the design variable constraints (see Section 5.3).

The figure shows a three-member frame with dimensions 100" and 100", supports at nodes 1 and 2, with loads P = 10,000 lb. and σ_f = 40,000 psi. at 1,2.

		Initial Values, in.	Optimum Values, in. Obtained Using	
			Exact Derivatives	Approximate Derivatives
$\eta_{max} = .1 \times 10^{-1}$ $\varepsilon_x = .1 \times 10^{-4}$	t_1	0.9	0.19816	0.19816
	b_1	9.0	10.00000	10.00000
	h_1	9.0	10.00000	10.00000
	t_2	0.9	0.10000	0.10000
	b_2	9.0	2.50000	2.50000
	h_2	9.0	2.50000	2.50000
	t_3	0.9	0.19816	0.19816
	b_3	9.0	10.00000	10.00000
	h_3	9.0	10.00000	10.00000
Volume, in.^{3}		8,748.00	1,649.85	1,649.85
Number of iterations			26	30
Computing time, sec.			84	100

TABLE 5.11 Optimum Design of a Three Member Frame Using CSDS

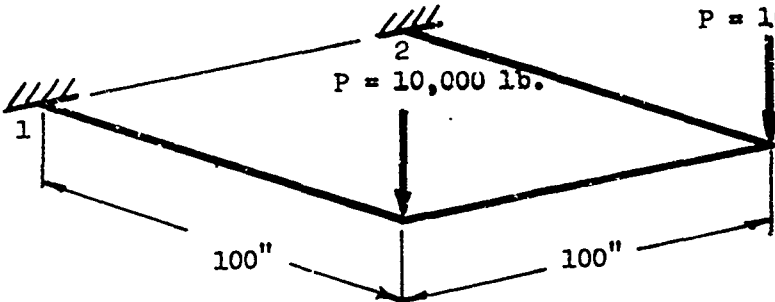|		Initial Values, in.	Optimum Values, in. Obtained Using	
			Exact Derivatives	Approximate Derivatives
$\eta_{max} = .5\text{x}10^{-2}$	t_1	0.9	0.10449	0.10408
$\epsilon_x = .1\text{x}10^{-4}$	b_1	9.0	9.02133	9.06760
	h_1	9.0	10.00000	10.00000
	t_2	0.9	0.10000	0.10000
	b_2	9.0	5.64220	5.64298
	h_2	9.0	7.87559	7.87507
	t_3	0.9	0.10449	0.20408
	b_3	9.0	9.02133	9.06760
	h_3	9.0	10.00000	10.00000
Volume, in³		8,748.00	1,052.66	1,051.52
Number of iterations			50 (limit)	50 (limit)
Computing time, sec.			93	93
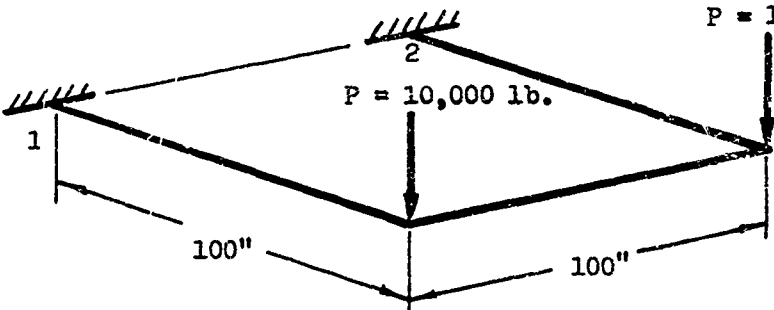
TABLE 5.12 Optimum Design of a Three Member Frame Using CSDS

TABLE 5.13 Optimum Design of a Three Member Frame Using CSDS

		Initial Values, in.	Optimum Values, in. Obtained Using	
			Exact Derivatives	Approximate Derivatives
η_{max} = 0.1	t_1	0.9	0.19908	0.19913
ε_x = .1x10^{-4}	b_1	9.0	10.00000	10.00000
	h_1	9.0	10.00000	10.00000
	t_2	0.9	0.10000	0.10000
	b_2	9.0	2.50000	2.50000
	h_2	9.0	2.50000	2.50000
	t_3	0.9	0.19908	0.19913
	b_3	9.0	10.00000	10.00000
	h_3	9.0	10.00000	10.00000
Volume, in^3		8,748.00	1,656.91	1,657.30
Number of iterations			16	78
Computing time, sec.			41	195

		Initial Values, in.	Optimum Values, in. Obtained Using	
			Exact Derivatives	Approximate Derivatives
$\eta_{max} = .1$	t_1	0.9	0.19908	0.19908
$\varepsilon_x = .1 \times 10^{-4}$	b_1	9.0	10.00000	10.00000
	h_1	9.0	10.00000	10.00000
	t_2	0.9	0.10000	0.10000
	b_2	9.0	2.50000	2.50000
	h_2	9.0	2.50000	2.50000
	t_3	0.9	0.19908	0.19908
	b_3	9.0	10.00000	10.00000
	h_3	9.0	10.00000	10.00000
Volume, in^3		8,748.00	1,656.91	1,656.91
Number of iterations			14	14
Computing time, sec.			48	48

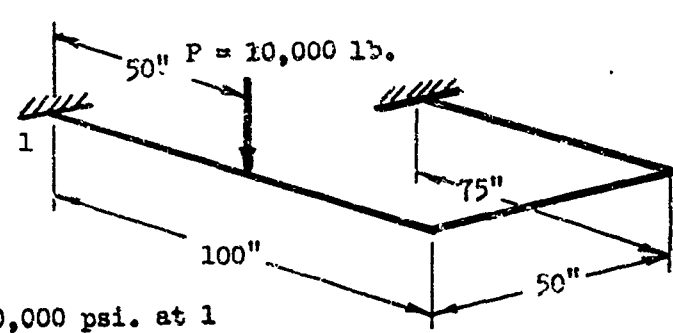TABLE 5.14   Optimum Design of a Three Member Frame Using CSDS

		Initial Values, in.	Optimum Values, in. Obtained Using	
			Exact Derivatives	Approximate Derivatives
$\eta_{max} = .1\times10^{-1}$	t_1	0.9	0.10000	0.10000
$\epsilon_x = .1\times10^{-4}$	b_1	9.0	9.53870	9.53870
	h_1	9.0	10.00000	10.00000
	t_2	0.1	0.10000	0.10000
	b_2	2.5	2.50000	2.50000
	h_2	2.5	2.50000	2.50000
	t_3	0.1	0.10000	0.10000
	b_3	3.0	2.50000	2.50000
	h_3	3.0	2.50000	2.50000
Volume, in³		3,051.00	506.77	506.77
Number of iterations			32	32
Computing time, sec.			80	80

TABLE 5.15   Optimum Design of a Three Member Frame Using CSDS

		Initial Values, in.	Optimum Values, in. Obtained Using	
			Exact Derivatives	Approximate Derivatives
$\eta_{max} = .2 \times 10^{-1}$	t_1	0.9	0.13394	0.13394
$\epsilon_x = .1 \times 10^{-4}$	b_1	9.0	10.00000	10.00000
	h_1	9.0	10.00000	10.00000
	t_2	0.9	1.00000	1.00000
	b_2	9.0	3.00000	3.00000
	h_2	9.0	3.00000	3.00000
	t_3	0.9	0.13394	0.13394
	b_3	9.0	10.00000	10.00000
	h_3	9.0	10.00000	10.00000
Volume, in^3		5,471.00	1,088.05	1,088.05
Number of iterations			35	35
Computing time, sec.			77	77
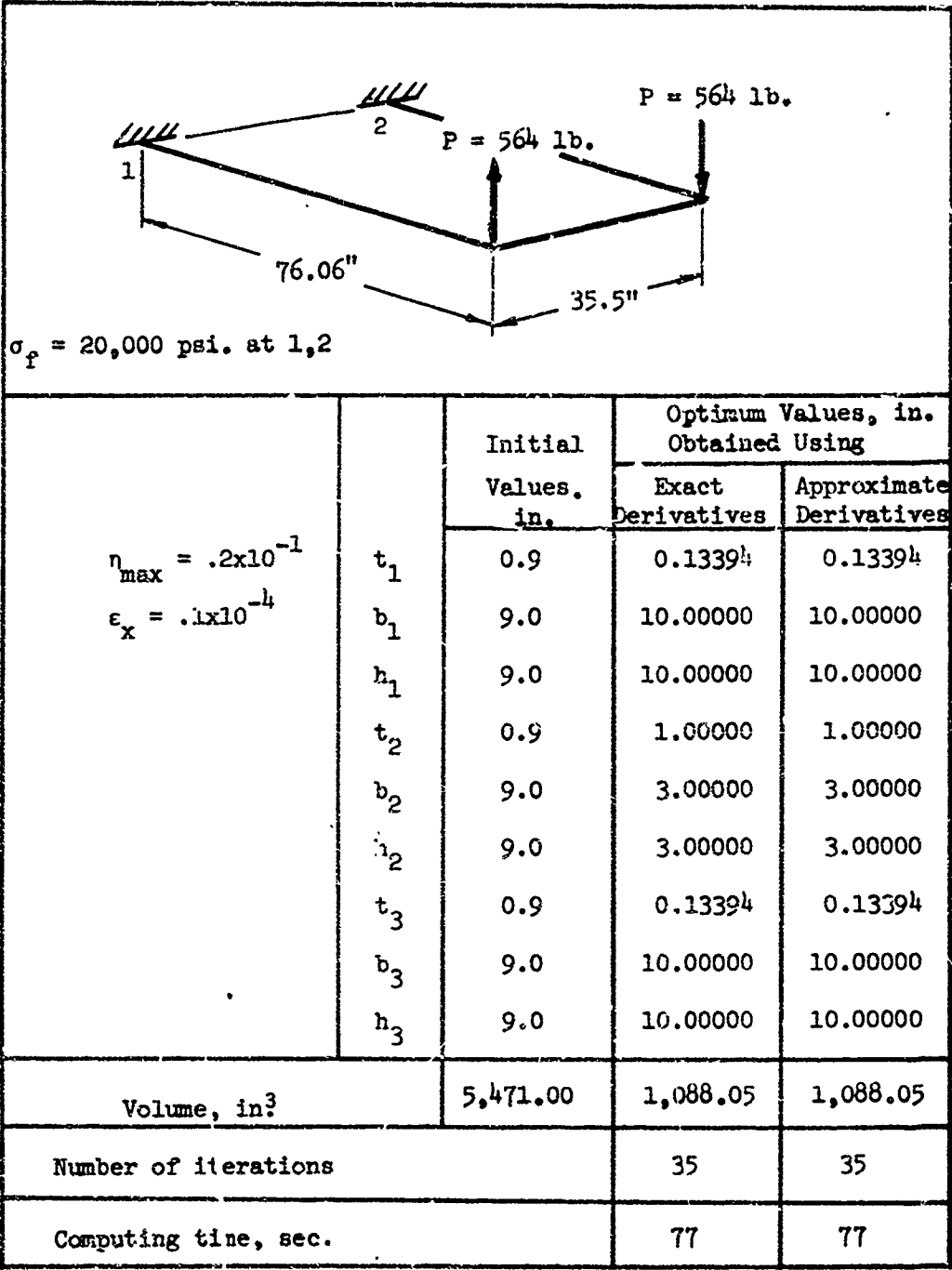
TABLE 5.16 Optimum Design of a Three Member Frame Using CSDS
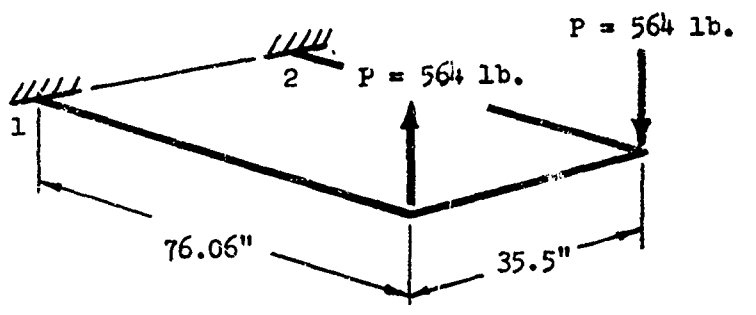
$\sigma_f = 20,000$ psi. at 1,2

		Initial Values, in.	Optimum Values, in. Obtained Using	
			Exact Derivatives	Approximate Derivatives
$\eta_{max} = .2 \times 10^{-1}$	t_1	0.9	0.20687	0.20687
$\epsilon_x = .1 \times 10^{-4}$	b_1	4.9	5.00000	5.00000
	h_1	11.0	12.00000	12.00000
	t_2	1.0	0.25000	0.25000
	b_2	10.0	3.00000	3.00000
	h_2	9.0	3.00000	3.00000
	t_3	0.9	0.20687	0.20687
	b_3	4.9	5.00000	5.00000
	h_3	11.0	12.00000	12.00000
Volume, in^3		5,067.81	1,141.52	1,141.52
Number of iterations			28	28
Computing time, sec.			70	70

TABLE 5.17 Optimum Design of a Three Member Frame Using CSDS

CHAPTER VI

CONCLUSIONS

The results of Chapter V show that a class of spatial structures can be optimally designed by the method of constrained steepest descent with state equations. In particular, this method appears to be superior to the sequential unconstrained minimization technique when necessary derivatives cannot be computed exactly. The class of problems which can be solved by the method is broad and includes those structures that can be analyzed by matrix displacement and finite element methods.

In large measure, the effectiveness of CSDS results from the fact that its formulation so closely matches the essential features of the original design problem. This is a more natural approach to the problem and leaves the associated functions in their most simple form. Other programming methods tend to complicate the formulation. In SUMT the surface created by adding a penalty term to the objective function exhibits characteristics that even the most robust unconstrained minimization techniques have difficulty coping with. In other approaches the state equations are used to write all of the constraints as functions of the design variables; as a consequence, complicated nonlinear functions are obtained. Another method defines the set of optimization variables to consist of the design variables

plus the state variables. For the structures considered in this research, minimization problems of dimension sixteen and twenty-seven would result. Such problems are formidable particularly when the minimization must be accomplished without calculating derivatives.

The method of constrained steepest descent with state equations avoids these pitfalls because the state equations and constraints involving the state variables are included directly in the formulation. Consequently, a great deal of flexibility is introduced into the solution since the functions are retained in their simplest form. Therefore, derivatives are approximated only when it is absolutely necessary since, in the case of simple functions, derivatives may be computed exactly without undue effort.

CSDS is an effective solution to the optimum structural design problem for two additional reasons. First of all, much of the "art" is removed, since only two parameters must be chosen — η_{max} and ε_x. In SUMT there are five — ε_{ex}, ε_p, ε_L, r_1, and C. Therefore, the amount of numerical experimentation that must be done for a given problem is greatly reduced. Secondly, the procedure has been written so that existing structural analysis techniques can be used to full advantage. Consequently, currently available algorithms based on matrix displacement and finite element techniques may be used in the optimization procedure with minimal effort for adaptation.

In this research, a basic configuration of the structure was first chosen and then the dimensions of individual members were

determined using methods of optimal design. There are some inter-
esting variations of this problem which may stimulate further inves-
tigation. These variations involve both the structure and the loading.

Structures of greater complexity can be generated by increasing
the number of the members or increasing the number of design vari-
ables per member. In the first case, the methods developed in this
research apply directly, since the basic form of the functions
involved is not changed. The incorporation of existing structural
analysis algorithms would be a natural way of coping with the
increased size of the problem. An example of the second case is the
introduction of linear variations of the height and width of the mem-
bers. A problem of this type will require an effective means of
determining the points of maximum stress in the structure and more
extensive numerical work in the structural analysis.

The optimum design problem can be formulated to include the
basic configuration of the structure by adding the lengths of the
members as design variables. This will increase the complexity of
the functions involved. For example, the objective function will be
a cubic in the design variables.

In practice, spatial structures may be subject to a variety of
randomly applied loads. In addition, the state variable constraints
may change for different loadings. An optimal design procedure for
spatial structures with multiple load sets and multiple constraint
sets should be developed for this type of problem.

The method of constrained steepest descent with state equations may have to be extended or modified to effectively solve some of these problems. For example, steepest descent methods converge slowly when the objective function has a long valley. Several methods have been developed to deal with problems of this type in unconstrained minimization and perhaps modifications similar to these can be introduced into CSDS. The method is certainly not limited to structural design, but may be applied to a wide variety of finite dimensional design problems. The magnitude and complexity of the problems that this method can successfully solve is not yet known and should be investigated further.

BIBLIOGRAPHY

[1] Bracken, J., and McCormick, G. P., *Selected Applications of Nonlinear Programming*, John Wiley and Sons, Inc., New York, 1968.

[2] Best, G., "A Method of Structural Weight Minimization Suitable for High-Speed Digital Computers," *AIAA Journal*, Vol. 1, No. 2, February 1963, pp. 478-479.

[3] Brown, D. M., and Ang, A. H., "Structural Optimization by Nonlinear Programming," *Journal of the Structural Division*, ASCE, Vol. 92, No. ST6, December 1966, pp. 319-340.

[4] Brown, D. M., and Ang, A. H., "Structural Optimization by Nonlinear Programming — Closure," *Journal of the Structural Division*, ASCE, Vol. 93, No. ST5, October 1967, pp. 618-619.

[5] Fiacco, A. V., and McCormick, G. P., *Nonlinear Programming — Sequential Unconstrained Minimization Techniques*, John Wiley and Sons, Inc., New York, 1968.

[6] Fletcher, R., "Function Minimization Without Evaluating Derivatives — A Review," *Computer Journal*, Vol. 8, No. 1, April 1965, pp. 33-41.

[7] Gellatly, R. A., and Gallagher, R. H., "A Procedure for Automated Minimum Weight Structural Design Part I: Theoretical Basis," *Aeronautical Quarterly*, Vol. 17, No. 3, August 1966, pp. 216-230.

[8] Gallatly, R. A., and Gallagher, R. H., "A Procedure for Automated Minimum Weight Structural Design Part II: Applications," *Aeronautical Quarterly*, Vol. 17, No. 4, November 1966, pp. 332-342.

[9] Gerard, George, "Optimum Structural Design Concepts for Aerospace Vehicles," *J. of Spacecraft and Rockets*, Vol. 3, No. 1, January 1966, pp. 5-18.

[10] Haug, E. J., "Optimal Design of Finite Dimensional Systems," from unpublished Lectures on Optimization, University of Iowa, Iowa City, Iowa, 1969.

[11] Langhaar, H. L., *Energy Methods in Applied Mechanics*, John Wiley and Sons, Inc., New York, 1962.

[12] Leon, A., "General-Purpose Optimization Procedures," Ph.D. Thesis, University of Michigan, 1965.

[13] Mel'ts, I. O., "Nonlinear Programming Methods for Optimizing Dynamical Systems in Function Space," _Automation and Remote Control_, No. 1, January 1968, pp. 68-73.

[14] Nicholls, Jack I., "Weight Minimization of Indeterminate Trusses Subjected to Single and Multiple Load Sets," Ph.D. Thesis, Purdue University, 1966.

[15] Pope, G. G., "The Design of Optimum Structures of Specified Basic Configuration," _Int. J. Mech. Sci._, Vol. 10, No. 4, April 1968, pp. 251-263.

[16] Powell, M. J. D., "An Efficient Method for Finding the Minimum of a Function of Several Variables Without Calculating Derivatives," _Computer Journal_, Vol. 7, No. 2, July 1964, pp. 155-162.

[17] Przemieniecki, J. S., _Theory of Matrix Structural Analysis_, McGraw-Hill Book Company, New York, 1968.

[18] Rubinstein, Moshe F., _Matrix Computer Analysis of Structures_, Prentice-Hall, Inc., Englewood Cliffs, N. J., 1966.

[19] Saaty, T. L., and Bram, J., _Nonlinear Mathematics_, McGraw-Hill Book Company, New York, 1964.

[20] Schmit, L. A., Kicher, T. P., and Morrow, W. M., "Structural Synthesis Capability for Integrally Stiffened Waffle Plates," _AIAA Journal_, Vol. 1, No. 12, December 1963, pp. 2820-2836.

[21] Schmit, L. A., and Fox, R. L., "An Integrated Approach to Structural Synthesis and Analysis," _AIAA Journal_, Vol. 3, No. 6, June 1965, pp. 1104-1112.

[22] Sheu, C. Y., and Prager, W., "Recent Developments in Optimal Structural Design," _Applied Mechanics Reviews_, Vol. 21, No. 10, October 1968, pp. 985-992.

[23] Toakley, A. R., "The Optimum Design of Triangulated Frameworks," _Int. J. Mech. Sci._, Vol. 10, No. 2, February 1968, pp. 115-127.

[24] Wasiutynski, Z., and Brandt, A., "The Present State of Knowledge in the Field of Optimum Design of Structures," _Applied Mech. Reviews_, Vol. 16, No. 5, May 1963, pp. 341-350.

[25] Wilde, D. J., and Beightler, C. S., _Foundations of Optimization_, Prentice-Hall, Inc., Englewood Cliffs, N. J., 1967.

[26] Wortman, John D., "FC.MIN (A FORAST Program to Find the Minimum of a Constrained Function Without Using Derivatives)," Technical Note No. 1677, Ballistic Research Laboratories, Aberdeen Proving Ground, Maryland, 1967.

[27] Zienkiewicz, O. C., _The Finite Element Method in Structural and Continuum Mechanics_, McGraw-Hill Book Company, New York, 1967.

APPENDIX

STRUCTURAL ANALYSIS

APPENDIX

STRUCTURAL ANALYSIS

A.1 The Basic Equations

The structural analysis used in this research follows a method discussed by Langhaar [11]. The analysis of a two member frame will be done in detail to illustrate the application of the method. The equations and associated matrices required for the analysis of the three member frame will then be shown.

Consider the two member frame in Figure A.1. The horizontal frame is composed of hollow rectangular members which are joined perpendicular to each other. The joint is rigid and transmits shear force as well as bending and twisting moments. The load, P, is applied normal to the frame at an arbitrary point along member (1,2). This point is denoted in Figure A.1 as point 4. The members are clamped at points 1 and 3. The reactions at these points are shown on the free body diagram in Figure A.2. The deformation of the frame is completely described by the displacements at joint 2. These displacements are the vertical deflection of point 2, and the rotations of the joint about the axes of the two members.

The behavior or state of the frame is specified by the value of the maximum failure stress occurring in the frame. This failure stress is calculated from the maximum distortion energy failure criterion using the following expression:
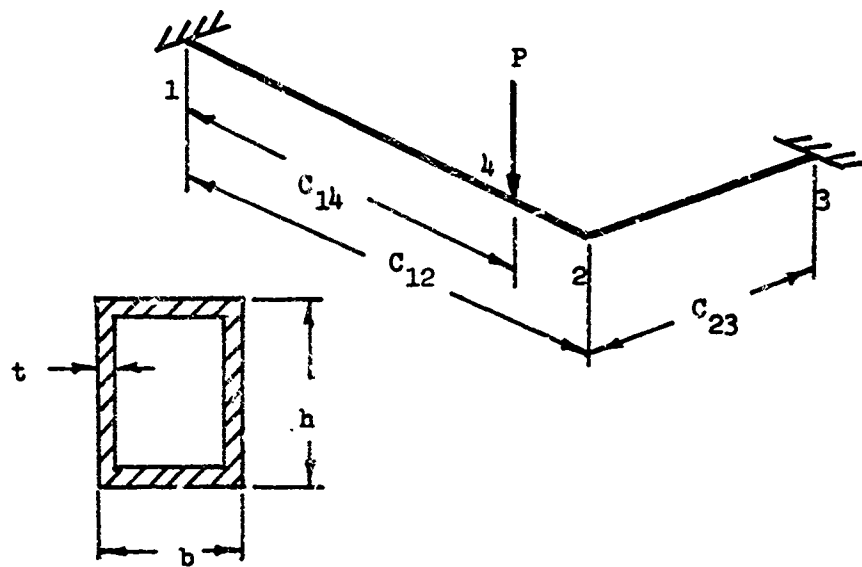
FIGURE A.1   Two Member Frame and Typical Member Cross-Section
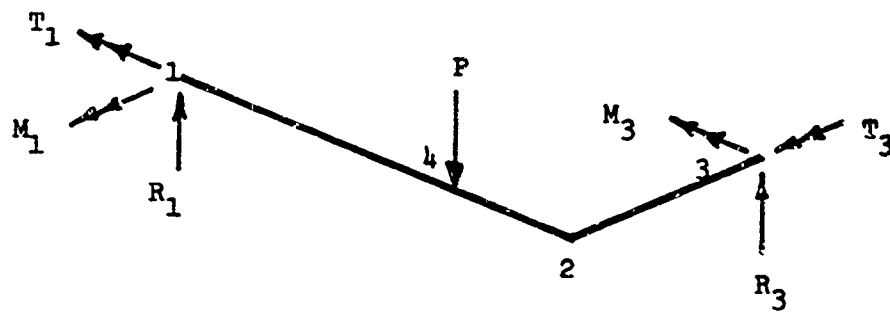

FIGURE A.2   Free Body Diagram for Two Member Frame

$$\sigma_f = \sqrt{\sigma^2 + 3\tau^2}, \tag{A-1}$$

where σ is the bending stress and τ is the shear stress which results from twisting the members. Twisting moments are applied only at the ends; therefore, the shear stress due to torsion is constant along the members. The bending moment varies along a member but will be a maximum either at the ends or under the applied load, since the cross-section is constant.

The failure stress will be a maximum when the bending stress is a maximum since the torsional shear stress is constant along a given member. Consequently, the maximum value of σ_f must occur at one of five points on the two member frame. The five possible points are the ends of each member and under the applied load. A free body diagram of member $(1,2)$ is shown on Figure A.3. The twisting moment at any section is

$$T = T_1 \tag{A-2}$$

and the bending moments at the three critical points are:

at point 1 $(x = 0)$

$$M = -M_1, \tag{A-3}$$

at point 4 $(x = C_{14})$

$$M = -M_1 + R_1 C_{14},$$

and at point 2 $(x = C_{12})$

$$M = -M_1 + R_1 C_{12} - P(C_{12} - C_{14}).$$

Similarly, for member $(2,3)$ (see Figure A.5), the twisting moment at

any cross-section is

$$T = T_3 \tag{A-4}$$

and the bending moments at the two critical points are:

at point 3 $(x = 0)$

$$M = -M_3, \tag{A-5}$$

and at point 2 $(x = C_{23})$

$$M = -M_3 + R_3 C_{23}. \tag{A-6}$$

The shear stress due to transverse loads has been neglected.

The bending stresses are computed from the equation

$$\sigma = \frac{M(h/2)}{I}, \tag{A-7}$$

where M is the bending moment, h is the height of the rectangular section, and I is the area moment of inertia of the section. Using the theory of torsion of thin tubes, the torsional shear stress is obtained as

$$\tau = \frac{T}{2At}, \tag{A-8}$$

where T is the twisting moment, A is the cross-sectional area, and t is the thickness of the tube. The stresses can be calculated using equations (A-7) and (A-8) if the end reactions of the members are known.

The frame member shown in Figure A.5 is subject to bending and twisting. The strain energy due to bending may be written in terms of the displacements of the ends of the member.
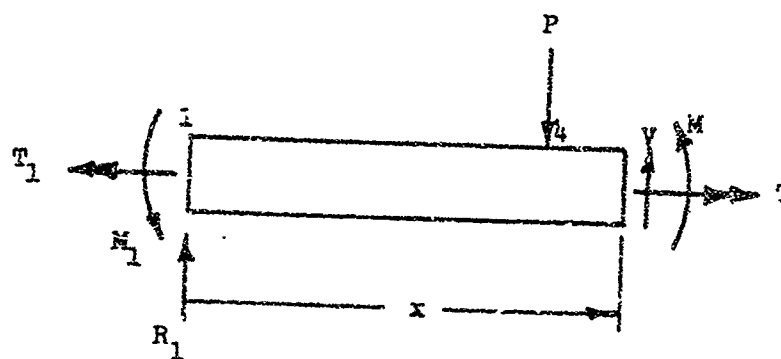
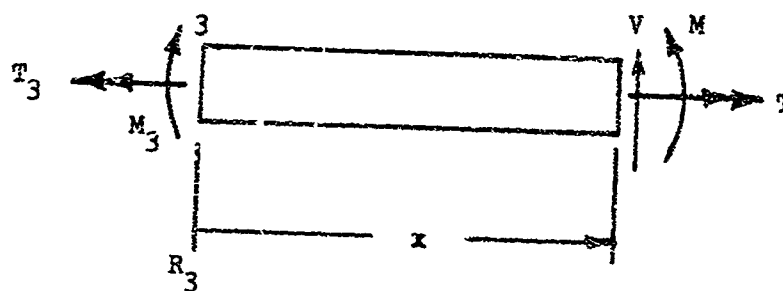FIGURE A.3 Free Body Diagram of Member (1,2)

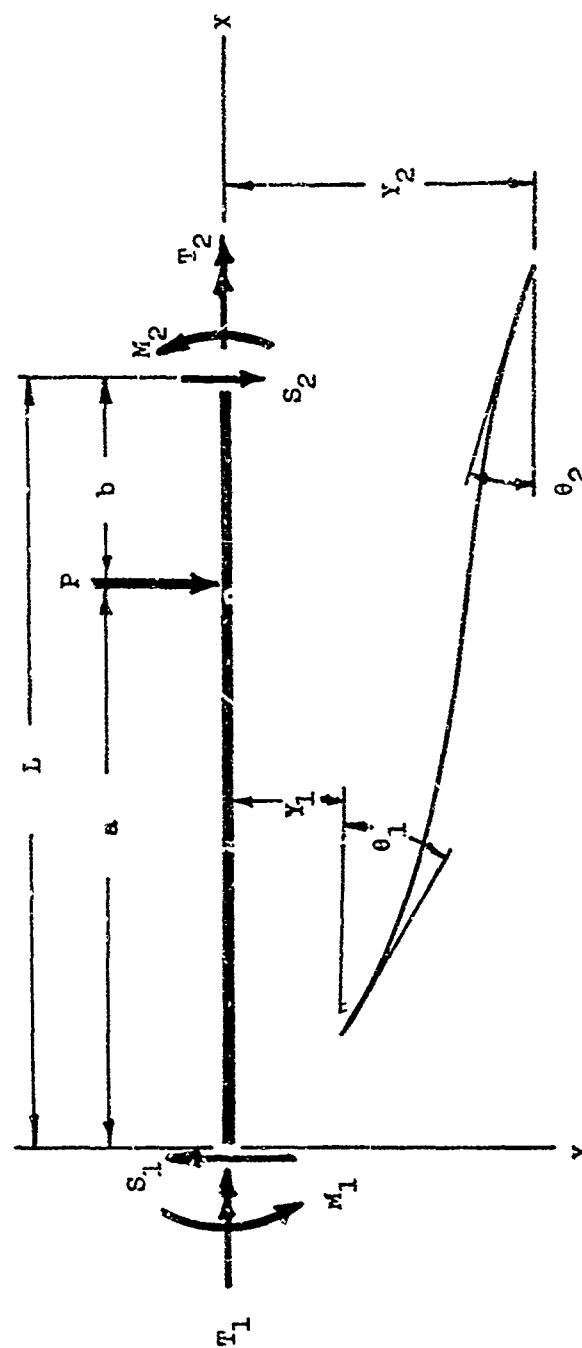FIGURE A.4 Free Body Diagram of Member (2,3)

FIGURE A.5 Frame Member Showing Reactions and End Displacements

$$U = K[\theta_1^2 + \theta_1\theta_2 + \theta_2^2 - 3\phi(\theta_1 + \theta_2) + 3\phi^2] + \text{const} \quad (A-9)$$

where $\phi = \dfrac{y_2 - y_1}{L}$ and $K = \dfrac{2EI}{L}$.

The strain energy due to torsion is

$$U = \frac{GJ}{2L} \alpha^2, \quad (A-10)$$

where α is the relative angle of twist between the ends of the member.

The potential energy of the external load is

$$\Omega = -\int_0^L p y \, dx, \quad (A-11)$$

where both p and y are functions of x. For the problems discussed herein p is a concentrated load. The deflection, y, may be written as a function of the end displacements as follows:

$$y = y_1 + \theta_1 x - (2\theta_1 + \theta_2 - 3\phi) \frac{x^2}{L} + (\theta_1 + \theta_2 - 2\phi) \frac{x^3}{L^2} + \text{const.} \quad (A-12)$$

Equations (A-9), (A-10), and (A-11) are used to obtain the total potential energy of the frame and the applied loads,

$$V = U_T + \Omega_T, \quad (A-13)$$

where U_T is the total strain energy, and Ω_T is the total potential energy of the external loads. By the principle of stationary potential energy, the required conditions for equilibrium are that

$$\frac{\partial V}{\partial q_i} = 0 \qquad i = 1, 2, \ldots n, \quad (A-14)$$

where the q_i are generalized coordinates (the three displacement

components at point 2). Equation (A-14) represents a set of n simultaneous linear equations in the n unknown generalized coordinates.

Once the deflections have been determined, the end reactions may be computed using the following:

$$M_1 = -K(2\theta_1 + \theta_2 - 3\phi) + \frac{Pab^2}{L^2} \qquad (A-15)$$

$$M_2 = -K(\theta_1 + 2\theta_2 - 3\phi) - \frac{Pa^2b}{L^2} \qquad (A-16)$$

$$S_1 = -\frac{3K}{L}(\theta_1 + \theta_2 - 2\phi) + \frac{Pb^2(L + 2a)}{L^3} \qquad (A-17)$$

$$\text{and} \quad S_2 = -\frac{3K}{L}(\theta_1 + \theta_2 - 2\phi) - \frac{Pa^2(L + 2b)}{L^3} . \qquad (A-18)$$

It should be noted that it is not necessary to write similar expressions for the twisting moments. Due to the relationship of action and reaction, the twisting moment in one member is just a bending moment in an adjacent member.

A.2 Application to Two Member Frames

The generalized coordinates are assigned as follows. The vertical displacement of point 2 is q_1, the joint rotation about the axis of member (2,3) is q_2, and the joint rotation about the axis of member (1,2) is q_3.

Using equation (A-11) the strain energy due to bending is

$$U = \frac{2E_{12}I_{12}}{C_{12}} \left[q_2^2 - 3 \frac{q_1}{C_{12}} q_2 + 3\left(\frac{q_1}{C_{12}}\right)^2 \right]. \qquad (A-19)$$

Using equation (A-2) the strain energy due to twisting may be written
as

$$U = \frac{G_{12}J_{12}}{2C_{12}} q_3^2. \tag{A-20}$$

For member (2,3) the strain energy due to bending is

$$U = \frac{2E_{23}I_{23}}{C_{23}} [q_3^2 - 3(\frac{q_1}{C_{23}})q_3 + 3(\frac{q_1}{C_{23}})^2], \tag{A-21}$$

and the strain energy due to twisting is

$$U = \frac{G_{23}J_{23}}{2C_{23}} q_2^2. \tag{A-22}$$

Equations (A-11) and (A-12) are used to obtain the potential energy
of the applied loads.

$$\Omega = -P(\frac{3C_{14}^2}{C_{12}^2} - \frac{2C_{14}^3}{C_{12}^3})q_1 - P(\frac{C_{14}^3}{C_{12}^2} - \frac{C_{14}^2}{C_{12}})q_2 \tag{A-23}$$

Adding the results of equations (A-19) through (A-23), the total
potential energy of the frame is

$$\tilde{V} = [\frac{6E_{12}I_{12}}{C_{12}^3} + \frac{6E_{23}I_{23}}{C_{23}^3}]q_1^2 + [\frac{2E_{12}I_{12}}{C_{12}} + \frac{G_{23}J_{23}}{2C_{23}}]q_2^2$$

$$+ [\frac{2E_{23}I_{23}}{C_{23}} + \frac{G_{12}J_{12}}{2C_{12}}]q_3^2 - \frac{6E_{12}I_{12}}{C_{12}^2} q_1 q_2 - \frac{6E_{23}I_{23}}{C_{23}^2} q_1 q_3$$

$$- P[\frac{3C_{14}^2}{C_{12}^2} - \frac{2C_{14}^3}{C_{12}^3}]q_1 - P[\frac{C_{14}^3}{C_{12}^2} - \frac{C_{14}^2}{C_{12}}]q_2. \tag{A-24}$$

When the principle of stationary potential energy is applied, three equations in q_1, q_2, and q_3 are obtained from equation (A-24)

$$\frac{\partial \bar{V}}{\partial q_1} = [\frac{12E_{12}I_{12}}{c_{12}^3} + \frac{12E_{23}I_{23}}{c_{23}^3}]q_1 - \frac{6E_{12}I_{12}}{c_{12}^2}q_2 - \frac{6E_{23}I_{23}}{c_{23}^2}q_3$$

$$- P[\frac{3c_{14}^2}{c_{12}^2} - \frac{2c_{14}^3}{c_{12}^3}] = 0 \qquad (A-25)$$

$$\frac{\partial \bar{V}}{\partial q_2} = - \frac{6E_{12}I_{12}}{c_{12}^2}q_1 + [\frac{4E_{12}I_{12}}{c_{12}} + \frac{G_{23}J_{23}}{c_{23}}]q_2$$

$$- P[\frac{c_{14}^3}{c_{12}^2} - \frac{c_{14}^2}{c_{12}}] = 0 \qquad (A-26)$$

$$\frac{\partial \bar{V}}{\partial q_3} = - \frac{6E_{23}I_{23}}{c_{23}^2}q_1 + [\frac{4E_{23}I_{23}}{c_{23}} + \frac{G_{12}J_{12}}{c_{12}}]q_3 = 0 \qquad (A-27)$$

These three equations may be written in matrix form.

$$AQ = P \qquad (A-28)$$

where:

$$A = \begin{bmatrix} \frac{12E_{12}I_{12}}{c_{12}^3} + \frac{12E_{23}I_{23}}{c_{23}^3} & - \frac{6E_{12}I_{12}}{c_{12}^2} & - \frac{6E_{23}I_{23}}{c_{23}^2} \\ - \frac{6E_{12}I_{12}}{c_{12}^2} & \frac{4E_{12}I_{12}}{c_{12}} + \frac{G_{23}J_{23}}{c_{23}} & 0 \\ - \frac{6E_{23}I_{23}}{c_{23}^2} & 0 & \frac{4E_{23}I_{23}}{c_{23}} + \frac{G_{12}J_{12}}{c_{12}} \end{bmatrix}$$

$$Q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

$$P = \begin{bmatrix} P\left(\dfrac{3C_{14}^2}{C_{12}^2} - \dfrac{2C_{14}^3}{C_{12}^3}\right) \\ \\ P\left(\dfrac{C_{14}^3}{C_{12}^2} - \dfrac{C_{14}^2}{C_{12}}\right) \\ \\ 0 \end{bmatrix}$$

Equations (A-7) through (A-10) are now used to obtain the end loads on the members. Only the loads at points 1 and 3 (see Figure A-2) are required to determine the stresses.

$$M_1 = -\frac{2E_{12}I_{12}}{C_{12}}\left[q_2 - 3\frac{q_1}{C_{12}}\right] + \frac{PC_{14}(C_{12} - C_{14})^2}{C_{12}^2} \qquad (A-29)$$

$$R_1 = -\frac{6E_{12}I_{12}}{C_{12}^2}\left[q_2 - 2\frac{q_1}{C_{12}}\right] + \frac{P(C_{12} - C_{14})^2(C_{12} + 2C_{14})}{C_{12}^3} \qquad (A-30)$$

$$T_1 = -\frac{2E_{23}I_{23}}{C_{23}}\left[2q_3 - 3\frac{q_1}{C_{23}}\right] \qquad (A-31)$$

$$M_3 = -\frac{2E_{23}I_{23}}{C_{23}}\left[q_3 - 3\frac{q_1}{C_{23}}\right] \qquad (A-32)$$

$$R_3 = -\frac{6E_{23}I_{23}}{C_{23}^2}\left[q_3 - 2\frac{q_1}{C_{23}}\right] \tag{A-33}$$

$$T_3 = -\frac{2E_{12}I_{12}}{C_{12}}\left[2q_2 - 3\frac{q_1}{C_{12}}\right] - \frac{PC_{14}^2(C_{12} - C_{14})}{C_{12}^2} \tag{A-34}$$

These equations may also be written in matrix form

$$M = BQ + F \tag{A-35}$$

The vector of joint displacements Q was defined previously. The matrix B and the vectors F and M are given below.

$$M = \begin{bmatrix} M_1 \\ R_1 \\ T_1 \\ M_3 \\ R_3 \\ T_3 \end{bmatrix}$$

$$F = \begin{bmatrix} \dfrac{PC_{14}(C_{12} - C_{14}^2)}{C_{12}^2} \\[2ex] \dfrac{P(C_{12} - C_{14})^2(C_{12} + 2C_{14})}{C_{12}^3} \\[2ex] 0 \\ 0 \\ 0 \\ -\dfrac{PC_{14}^2(C_{12} - C_{14})}{C_{12}^2} \end{bmatrix}$$

$$B = \begin{bmatrix} \dfrac{6E_{12}I_{12}}{c_{12}^2} & -\dfrac{2E_{12}I_{12}}{c_{12}} & 0 \\[3mm] \dfrac{12E_{12}I_{12}}{c_{12}^3} & -\dfrac{6E_{12}I_{12}}{c_{12}^2} & 0 \\[3mm] \dfrac{6E_{23}I_{23}}{c_{23}^2} & 0 & -\dfrac{4E_{23}I_{23}}{c_{23}} \\[3mm] \dfrac{6E_{23}I_{23}}{c_{23}^2} & 0 & -\dfrac{2E_{23}I_{23}}{c_{23}} \\[3mm] \dfrac{12E_{23}I_{23}}{c_{23}^2} & 0 & -\dfrac{6E_{23}I_{23}}{c_{23}^2} \\[3mm] \dfrac{6E_{12}I_{12}}{c_{12}^2} & -\dfrac{4E_{12}I_{12}}{c_{12}^2} & 0 \end{bmatrix}$$

The stresses are computed using equations (A-7) and (A-8). As can be seen from these equations, and equations (A-2) through (A-6), the stresses will be linear functions of the end reactions. Expressed in matrix form the stresses may be computed from the following equation:

$$S = CM + \tilde{P}, \tag{A-36}$$

where M is the vector of end reactions defined previously. The matrix C and the vectors $\tilde{P}$ and S are defined below.

$$
C = \begin{bmatrix}
\dfrac{h_{12}}{2I_{12}} & 0 & 0 & 0 & 0 & 0 \\[2ex]
-\dfrac{h_{12}}{2I_{12}} & 0 & 0 & 0 & \dfrac{h_{12}C_{12}}{2I_{12}} & 0 \\[2ex]
-\dfrac{h_{12}}{2I_{12}} & 0 & 0 & 0 & \dfrac{h_{12}C_{12}}{2I_{12}} & 0 \\[2ex]
0 & 0 & \dfrac{1}{2A_{12}t_{12}} & 0 & 0 & 0 \\[2ex]
0 & 0 & 0 & \dfrac{h_{23}}{2I_{23}} & 0 & -\dfrac{h_{23}C_{23}}{2I_{23}} \\[2ex]
0 & 0 & 0 & \dfrac{h_{23}}{2I_{23}} & 0 & 0 \\[2ex]
0 & \dfrac{1}{2A_{23}t_{23}} & 0 & 0 & 0 & 0
\end{bmatrix}
$$

$$
\tilde{P} = \begin{bmatrix}
0 \\[1ex]
-\dfrac{Ph_{12}(C_{12} - C_{14})}{2I_{12}} \\[2ex]
0 \\[1ex]
0 \\[1ex]
0 \\[1ex]
0 \\[1ex]
0
\end{bmatrix}
\qquad
S = \begin{bmatrix}
S_1 \\
S_2 \\
S_3 \\
S_4 \\
S_5 \\
S_6 \\
S_7
\end{bmatrix}
$$

The components S_1, S_2, and S_3 are the bending stresses at points 1, 2, and 4, respectively, on member (1,2). S_5 and S_6 are the bending

stresses on member $(2,3)$ at points 2 and 3. S_4 and S_7 are the shearing stresses due to torsion in member $(1,2)$ and $(2,3)$.

A.3 Application to Three Member Frames

Equations $(A-1)$ through $(A-18)$ may be applied to the three member frame by following the procedures outlined in Section A.2. The resulting matrix equations, $(A-28)$, $(A-35)$, and $(A-36)$, are repeated here.

$$AQ = P \qquad\qquad\qquad (A-28)$$

$$M = BQ + F \qquad\qquad\qquad (A-35)$$

$$S = CM + \tilde{P} \qquad\qquad\qquad (A-36)$$

These equations may be used directly by defining the necessary vectors and matrices for the three member frame shown in Figure A.6.

The deflections, Q, are defined as follows:

q_1, is the vertical displacement of joint 3;

q_2, the rotation of joint 3 about axis $(3,4)$;

q_3, the rotation of joint 3 about axis $(1,3)$;

q_4, the vertical displacement of joint 5;

q_5, the rotation of joint 5 about axis $(3,5)$;

and q_6, the rotation of joint 5 about axis $(5,7)$.

The vectors M and S are defined using the free body diagram in Figure A.7.
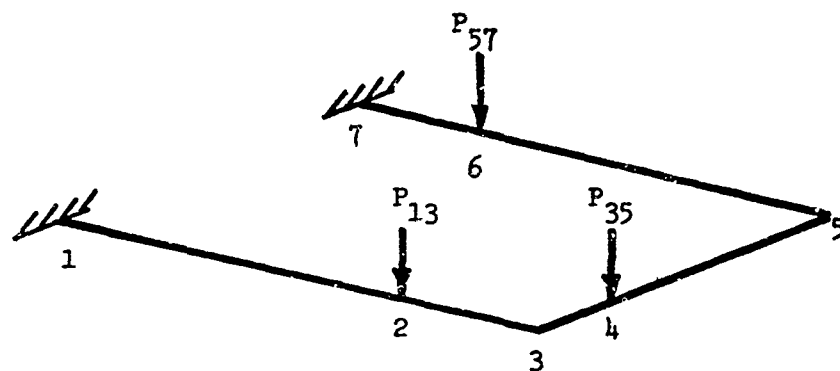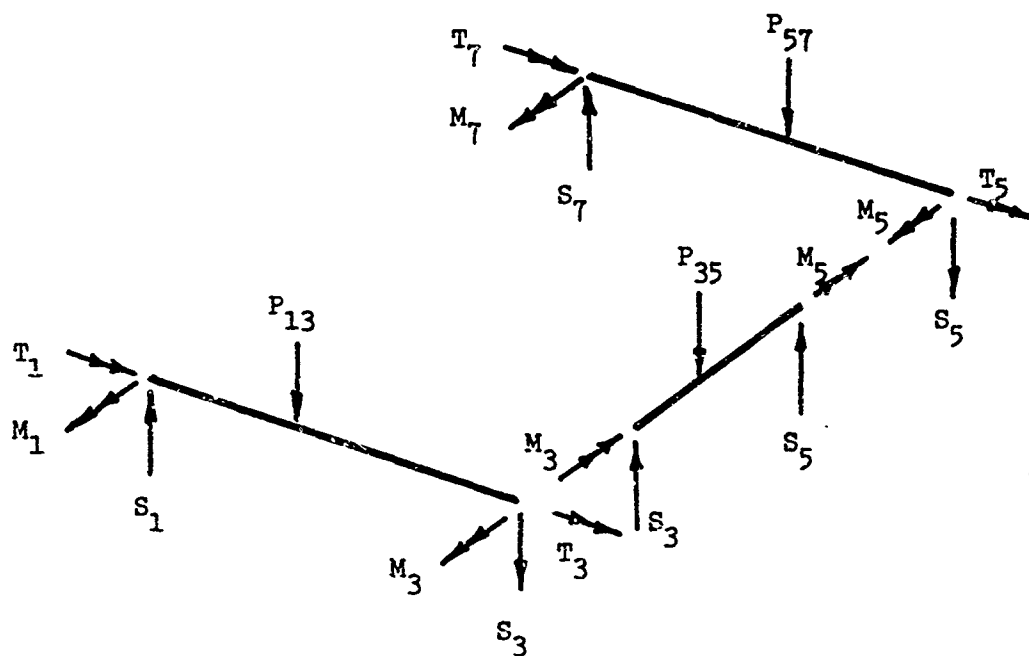
FIGURE A.6   Three Member Frame

FIGURE A.7   Free Body Diagram of the
Three Member Frame

$$M = \begin{bmatrix} M_1 \\ M_3 \\ M_7 \\ T_3 \\ T_5 \\ S_1 \\ S_3 \\ S_7 \end{bmatrix}$$

and

$$S = \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \vdots \\ \sigma_{12} \end{bmatrix},$$

where

σ_1, is the bending stress at point 1 on member $(1,3)$;

σ_2, the bending stress at point 2 in member $(1,3)$;

σ_3, the bending stress at point 3 in member $(1,3)$;

σ_4, the torsional stress in member $(1,3)$;

σ_5, the bending stress at point 7 in member $(5,7)$;

σ_6, the bending stress at point 6 in member $(5,7)$;

σ_7, the bending stress at point 5 in member $(5,7)$;

σ_8, the torsional stress in member $(5,7)$;

σ_9, the bending stress at point 3 in member (3,5);

σ_{10}, the bending stress at point 4 in member (3,5);

σ_{11}, the bending stress at point 5 in member (3,5);

and σ_{12}, the torsional stress in member (3,5).

The other matrices and vectors required for equations (A-28), (A-35), and (A-36), follow. The notation C_{ij} denotes the distance between point i and point j on the frame.

$$
P = \begin{bmatrix}
P_{13}[\dfrac{3c_{12}^2}{c_{13}^2} - \dfrac{2c_{12}^3}{c_{13}^3}] + P_{35}[1 - \dfrac{3c_{34}^2}{c_{35}^2} + \dfrac{2c_{34}^3}{c_{35}^3}] \\[2em]
P_{13}[\dfrac{c_{12}^3}{c_{13}^2} - \dfrac{c_{12}^2}{c_{13}}] \\[2em]
P_{35}[c_{34} - \dfrac{2c_{34}^2}{c_{35}} + \dfrac{c_{34}^3}{c_{35}^2}] \\[2em]
P_{35}[\dfrac{3c_{34}^2}{c_{35}^2} - \dfrac{2c_{34}^3}{c_{35}^3}] + P_{57}[\dfrac{3c_{76}^2}{c_{57}^2} - \dfrac{2c_{76}^3}{c_{57}^3}] \\[2em]
P_{57}[\dfrac{c_{76}^3}{c_{57}^2} - \dfrac{c_{76}^2}{c_{57}}] \\[2em]
P_{35}[\dfrac{c_{34}^3}{c_{35}^2} - \dfrac{c_{34}^2}{c_{35}}]
\end{bmatrix}
$$

$$
A =
\begin{bmatrix}
12EI\left[\dfrac{I_{13}}{c_{13}^3} + \dfrac{I_{35}}{c_{35}^3}\right] & -\dfrac{6EI_{13}}{c_{13}^2} & -\dfrac{6EI_{35}}{c_{35}^2} & -\dfrac{12EI_{35}}{c_{35}^3} & 0 & \dfrac{6EI_{35}}{c_{35}^2} \\[3ex]
-\dfrac{6EI_{13}}{c_{13}^2} & \dfrac{4EI_{13}}{c_{13}} + \dfrac{GJ_{35}}{c_{35}} & 0 & 0 & -\dfrac{GJ_{35}}{c_{35}} & 0 \\[3ex]
\dfrac{6EI_{35}}{c_{35}^2} & 0 & \dfrac{4EI_{35}}{c_{35}} + \dfrac{GJ_{13}}{c_{13}} & -\dfrac{6EI_{35}}{c_{35}^2} & 0 & \dfrac{2EI_{35}}{c_{35}} \\[3ex]
-\dfrac{12EI_{35}}{c_{35}^3} & 0 & -\dfrac{6EI_{35}}{c_{35}^2} & 12EI\left[\dfrac{I_{35}}{c_{35}^3} + \dfrac{I_{57}}{c_{57}^3}\right] & -\dfrac{6EI_{57}}{c_{57}^2} & -\dfrac{6EI_{35}}{c_{35}^2} \\[3ex]
0 & -\dfrac{GJ_{35}}{c_{35}} & 0 & -\dfrac{6EI_{57}}{c_{57}^2} & \dfrac{4EI_{57}}{c_{57}} + \dfrac{GJ_{35}}{c_{35}} & 0 \\[3ex]
\dfrac{6EI_{35}}{c_{35}^2} & 0 & \dfrac{2EI_{35}}{c_{35}} & -\dfrac{6EI_{35}}{c_{35}^2} & 0 & \dfrac{4EI_{35}}{c_{35}} + \dfrac{GJ_{57}}{c_{57}}
\end{bmatrix}
$$

108

$$B = \begin{bmatrix}
\dfrac{6EI_{13}}{c_{13}^2} & -\dfrac{2EI_{13}}{c_{13}} & 0 & 0 & 0 & 0 \\[2ex]
\dfrac{6EI_{13}}{c_{13}^2} & -\dfrac{4EI_{13}}{c_{13}} & 0 & 0 & 0 & 0 \\[2ex]
0 & 0 & 0 & \dfrac{6EI_{57}}{c_{57}^2} & -\dfrac{2EI_{57}}{c_{57}} & 0 \\[2ex]
\dfrac{6EI_{35}}{c_{35}^2} & 0 & \dfrac{4EI_{35}}{c_{35}} & -\dfrac{6EI_{35}}{c_{35}^2} & 0 & \dfrac{2EI_{35}}{c_{35}} \\[2ex]
\dfrac{6EI_{35}}{c_{35}^2} & 0 & \dfrac{2EI_{35}}{c_{35}} & -\dfrac{6EI_{35}}{c_{35}^2} & 0 & \dfrac{4EI_{35}}{c_{35}} \\[2ex]
\dfrac{12EI_{13}}{c_{13}^3} & -\dfrac{6EI_{13}}{c_{13}^2} & 0 & 0 & 0 & 0 \\[2ex]
\dfrac{12EI_{13}}{c_{13}^3} & -\dfrac{6EI_{13}}{c_{13}^2} & 0 & 0 & 0 & 0 \\[2ex]
0 & 0 & 0 & \dfrac{12EI_{57}}{c_{57}^3} & \dfrac{6EI_{57}}{c_{57}^2} & 0
\end{bmatrix}$$

$$
F = \begin{bmatrix}
\dfrac{P_{13}C_{12}(C_{13} - C_{12})^2}{C_{13}^2} \\[2ex]
-\dfrac{P_{13}C_{12}^2(C_{13} - C_{12})}{C_{13}^2} \\[2ex]
\dfrac{P_{57}C_{56}^2(C_{57} - C_{56})}{C_{57}^2} \\[2ex]
-\dfrac{P_{35}C_{34}(C_{35} - C_{34})^2}{C_{35}^2} \\[2ex]
\dfrac{P_{35}C_{34}^2(C_{35} - C_{34})}{C_{35}^2} \\[2ex]
\dfrac{P_{13}(C_{13} - C_{12})^2(C_{13} + 2C_{12})}{C_{13}^3} \\[2ex]
-\dfrac{P_{13}C_{12}^2(C_{13} + 2C_{32})}{C_{13}^3} \\[2ex]
\dfrac{P_{57}C_{56}^2(C_{57} + 2C_{76})}{C_{57}^3}
\end{bmatrix}
$$

$$
C = \begin{bmatrix}
\dfrac{x_3}{2I_{13}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\[2ex]
\dfrac{x_3}{2I_{13}} & 0 & 0 & 0 & 0 & -\dfrac{x_3 C_{12}}{2I_{13}} & 0 & 0 \\[2ex]
\dfrac{x_3}{2I_{13}} & 0 & 0 & 0 & 0 & -\dfrac{C_{13} x_3}{2I_{13}} & 0 & 0 \\[2ex]
0 & 0 & 0 & \dfrac{1}{2A_{13} x_1} & 0 & 0 & 0 & 0 \\[2ex]
0 & 0 & \dfrac{x_9}{2I_{57}} & 0 & 0 & 0 & 0 & 0 \\[2ex]
0 & 0 & \dfrac{x_9}{2I_{57}} & 0 & 0 & 0 & 0 & -\dfrac{x_9 C_{76}}{2I_{57}} \\[2ex]
0 & 0 & \dfrac{x_9}{2I_{57}} & 0 & 0 & 0 & 0 & -\dfrac{x_9 C_{57}}{2I_{57}} \\[2ex]
0 & 0 & 0 & 0 & \dfrac{1}{2A_{57} x_7} & 0 & 0 & 0 \\[2ex]
0 & 0 & 0 & \dfrac{x_6}{2I_{35}} & 0 & 0 & 0 & 0 \\[2ex]
0 & 0 & 0 & \dfrac{x_6}{2I_{35}} & 0 & 0 & \dfrac{C_{34} x_6}{2I_{35}} & 0 \\[2ex]
0 & 0 & 0 & \dfrac{x_6}{2I_{35}} & 0 & 0 & \dfrac{C_{35} x_6}{2I_{35}} & 0 \\[2ex]
0 & \dfrac{1}{2A_{35} x_4} & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

$$\tilde{P} = \begin{bmatrix} 0 \\[6pt] 0 \\[6pt] \dfrac{P_{13}(C_{13} - C_{12})x_3}{2I_{13}} \\[10pt] 0 \\[6pt] 0 \\[6pt] 0 \\[6pt] \dfrac{P_{57}(C_{57} - C_{76})x_9}{2I_{57}} \\[10pt] 0 \\[6pt] 0 \\[6pt] 0 \\[6pt] \dfrac{-P_{35}(C_{35} - C_{34})x_6}{2I_{35}} \end{bmatrix}$$

DOCUMENT CONTROL DATA - R & D .

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1 ORIGINATING ACTIVITY (Corporate author)	2a. REPORT SECURITY CLASSIFICATION
The University of Iowa	Unclassified
	2b. GROUP NA

3. REPORT TITLE

OPTIMUM DESIGN OF SPATIAL STRUCTURES

4. DESCRIPTIVE NOTES *(Type of report and inclusive dates)*

Technical Report

5. AUTHOR(S) *(First name, middle initial, last name)*

Donald L. Bartel

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
August 1969	122	27
8a. CONTRACT OR GRANT NO. DAAF03-69-C-0014	9a. ORIGINATOR'S REPORT NUMBER(S)	
b. PROJECT NO.	Project Themis Report No. 12	
c.	9b. OTHER REPORT NO(S) *(Any other numbers that may be assigned this report)*	
d.		

10. DISTRIBUTION STATEMENT

Distribution of this report is unlimited

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
	U.S. Army Weapons Command R*E Directorate Rock Island, Illinois 61201

13. ABSTRACT

This research presents a systematic approach to the optimal design of spatial structures for minimum weight subject to constraints on stress and geometry. The optimization procedures discussed are general and may be applied to structures which can be analyzed by matrix displacement or finite element methods.

Two methods of mathematical programming are applied to obtain a minimum weight design. The first is the sequential unconstrained minimization technique (SUMT), and the second is the method of constrained steepest descent with state equations (CSDS). Both of these techniques require derivatives of the objective and constraint functions to improve estimates of the optimum design. In many structural problems, it is very difficult or impossible to compute these derivatives exactly; existing structural analysis algorithms are generally not equipped to compute these derivatives. In order to take full advantage of existing analysis capability, the programming techniques in this research have been developed assuming that such derivatives are not available.

Optimal structural design problems are characterized by an objective function (the weight), state variables (the stresses and deflections), design variables, state equations (the structural analysis), and constraints which may be functions of the design and state variables. When the state equations are used to write all of the constraints as functions of the design variables, a nonlinear programming problem results. The sequential unconstrained minimization technique reduces the constrained nonlinear programming problem to a sequence of unconstrained

DD `FORM` **1473** REPLACES DD FORM 1473, 1 JAN 64, WHICH IS OBSOLETE FOR ARMY USE.

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Optimum design,						
minimum weight,						
structures,						
spatial structures,						
mathematical programming,						
spatial frames.						

problems which can be solved using existing unconstrained minimization techniques. A SUMT program was written for this research using Powell's method of unconstrained minimization without derivatives. The required minimization of a function along a line uses a combination of a Fibonacci search (to bracket the minimum) and a quadratic approximation of the minimum.

The method of constrained steepest descent differs from the usual nonlinear programming problem in that the state equations and the state variable constraints appear explicitly in the formulation. This provides a natural matching of the essential features of the design problem and the method used to obtain its solution. The design problem is linearized about a candidate design and the desired improvement in the design variables, δx, is required to be small by demanding that $\delta x^T w^{-1} \delta x = \xi^2$, where ξ is a small number and w is a positive definite weighting matrix. The Kuhn-Tucker necessary conditions are then applied to the resulting nonlinear problem. As a direct consequence, δx is specified in terms of two components; δx_1 which reduces the objective function consistent with the constraints, and δx_2 which directs the search for a minimum back to the feasible region if constraints have been violated. The method was applied using both exact and approximate derivatives, so that its effectiveness when derivatives are not available could be assessed.

A spatial structures which occurs frequently in practice is the plane frame with out-of-plane loads. Although such structures are generally made up of relatively few members, they may have many design variables since several design parameters must be specified for each member. The programming methods were applied to a number of two and three member frames of this type. From the results, it appears that CSDS has significant advantages over SUMT both in terms of computational time and the number of times that candidate designs must be analyzed. The results also show that CSDS performs as well when derivatives are approximated as it does when they can be computed exactly. The effectiveness of SUMT is reduced significantly if the derivatives are unavailable.

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY *(Corporate author)*	2a. REPORT SECURITY CLASSIFICATION
University of Iowa	Unclassified
	2b. GROUP
	NA

3. REPORT TITLE

OPTIMUM DESIGN OF SPATIAL STRUCTURES

4. DESCRIPTIVE NOTES *(Type of report and inclusive dates)*

Technical Report

5. AUTHOR(S) *(First name, middle initial, last name)*

Donald L. Bartel

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
August 1969	125	27

8a. CONTRACT OR GRANT NO.	9a. ORIGINATOR'S REPORT NUMBER(S)
DAAF03-69-C-0014	
b. PROJECT NO.	Project Themis Report No. 12
c.	9b. OTHER REPORT NO(S) *(Any other numbers that may be assigned this report)*
d.	

10. DISTRIBUTION STATEMENT

Distribution of this report is unlimited.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
	U. S. Army Weapons Command
	R & E Directorate
	Rock Island, Illinois 61201

13. ABSTRACT

A spatial structure which occurs frequently in practice is the plane frame with out-of-plane loads. Although such structures are generallly made up of relatively few members, they may have many design variables since several design parameters must be specified for each member. The programming methods were applied to a number of two and three member frames of this type. From the results, it appears that CSDS has significant advantages over SUMT both in terms of computational time and the number of times that candidate designs must be analyzed. The results also show that CSDS performs as well when derivatives are approximated as it does when they can be computed exactly. The effectiveness of SUMT is reduced significantly if the derivatives are unavailable.

14.

KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
optimum design minimum weight structures spatial structures mathematical programming spatial frames						